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POINCARÉ SERIES OF RESOLUTIONS OF SURFACE SINGULARITIES

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Abstract. Let $X \to \mathrm{spec}(R)$ be a resolution of singularities of a normal surface singularity $\mathrm{spec}(R)$, with integral exceptional divisors $E_1, \ldots, E_r$. We consider the Poincaré series

$$g = \sum_{\underline{n} \in \mathbf{N}^r} h(\underline{n}) t^{\underline{n}},$$

where

$$h(\underline{n}) = \ell(R/\Gamma(X, \mathcal{O}_X(-n_1 E - 1 - \cdots - n_r E_r)).$$

We show that if R/m has characteristic zero and $\mathrm{Pic}^0(X)$ is a semi-abelian variety, then the Poincaré series g is rational. However, we give examples to show that this series can be irrational if either of these conditions fails.

1. Introduction

Suppose that R is an excellent, normal local ring of dimension 2, with maximal ideal m. Let

$$f : X \to \mathrm{spec}(R)$$

be a resolution of singularities, with integral exceptional divisors $E_1, \ldots, E_r$.

If $\underline{n} = (n_1, \ldots, n_r) \in \mathbf{N}^r$, let

$$D_{\underline{n}} = \sum_{i=1}^{r} n_i E_i.$$

$\Gamma(X, \mathcal{O}_X(-D_{\underline{n}})) \subset R$ is an ideal, and $R/\Gamma((X, \mathcal{O}_X(-D_{\underline{n}}))$ has finite length as an R-module. Consider the function

$$h(\underline{n}) = \ell(R/\Gamma((X, \mathcal{O}_X(-D_{\underline{n}})))$$

and the Poincaré series

$$g = \sum_{\underline{n} \in \mathbf{N}^r} h(\underline{n}) t^{\underline{n}},$$

where $t^{\underline{n}} = t_1^{n_1} \ldots t_r^{n_r}$ is a monomial in the variables $t_1, \ldots, t_r$.

By the local Riemann Roch theorem (cf. (6))

$$h(\underline{n}) = (\text{quadratic polynomial in } \underline{n}\ ) - h^1(X, \mathcal{O}_X(-D_{\underline{n}})). \tag{1}$$

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Thus the Poincaré series

$$g = (\text{rational series in } \underline{n}\ ) - \sum_{\underline{n}\in\mathbf{N}^r} h^1(X, \mathcal{O}_X(-D_{\underline{n}}))t^{\underline{n}}. \tag{2}$$

In this paper we consider the form of the function $h(\underline{n})$, and the question of the rationality of g.

If $r = 1$, the situation is very simple, as $-D_1$ is ample, so that

$$h(n_1) = \text{ quadratic polynomial in } n_1 \text{ for } n_1 >> 0$$

and g is thus rational. In this case ($r = 1$) $\bigoplus_{n\geq 0}\Gamma(X, \mathcal{O}_X(-n_1D_1))$ is a finitely generated R-algebra, so $h(n_1)$ is a Hilbert polynomial (for $n_1 >> 0$).

If $r > 1$, the behavior of $h(\underline{n})$ is much more subtle. We first observe that if ($\mathbf{k} = R/m$ is algebraically closed of characteristic zero and) R is not a rational singularity, then there exists a resolution $f : X \to \text{spec}(R)$ such that $\bigoplus_{\underline{n}\in\mathbf{N}^r}\Gamma(X, \mathcal{O}_X(-D_{\underline{n}}))$ is not a finitely generated R-algebra, so we might not expect polynomial-like behavior of $h(\underline{n})$, or rationality of the Poincaré series g. To see this, we first observe that with our assumptions, R has a rational singularity if and only if the divisor class group $\text{Cl}(R)$ of R is a torsion group (as follows from [2] and [19]). By Theorem 4 of [8], if $\text{Cl}(R)$ is not torsion, then there exist a resolution $f : X \to \text{spec}(R)$ and an exceptional divisor F on X such that $\bigoplus_{n\geq 0}\Gamma(X, \mathcal{O}_X(-nF))$ is not a finitely generated R-algebra. Thus the ring $\bigoplus_{\underline{n}\in\mathbf{N}^r}\Gamma(X, \mathcal{O}_X(-D_{\underline{n}}))$ is not a finitely generated R-algebra.

If $r > 1$, then even in the best cases $h^1(X, \mathcal{O}_X(-D_{\underline{n}}))$ is a complicated function. If $-D_{\underline{n}}$ is sufficiently ample, then $h^1(X, \mathcal{O}_X(-D_{\underline{n}}))$ will vanish. More generally, $h^1(X, \mathcal{O}_X(-D_{\underline{n}}))$ will tend to be small if $D_{\underline{n}}$ is in the cone $\mathbf{E}^+$ of exceptional curves D such that $-D$ is nef ($(D \cdot E_i) \leq 0$ for all i). To be precise, $h^1(X, \mathcal{O}_X(-D_{\underline{n}}))$ is bounded for $D_{\underline{n}} \in \mathbf{E}^+$ (as follows from Lemma 2.4). On the other hand, $h^1(X, \mathcal{O}_X(-D_{\underline{n}}))$ will tend to be large if $D_{\underline{n}}$ is far outside of $\mathbf{E}^+$.

We prove (proposition 6.3) that there exists an "abstract complex of polyhedral sets" $\mathcal{P}$, whose union is $\mathbf{Q}^r_{\geq 0}$, such that, for $P \in \mathcal{P}$ and $\underline{n} \in P \cap \mathbf{N}^r$,

$$h(\underline{n}) = Q(\underline{n}) + L(\underline{n}) + \varphi(\underline{n}),$$

where $Q(\underline{n})$ is a quadratic polynomial, $L(\underline{n})$ is a linear function with periodic coefficients (that is, $L(\underline{n}) = \sum_{i=1}^r \varphi_i(\underline{n})n_i$ where the $\varphi_i(\underline{n})$ are periodic functions), and $\varphi(\underline{n})$ is a bounded function.

To construct $\mathcal{P}$ we first construct a fan Σ subdividing $\mathbf{Q}^r_{\geq 0}$ on which the Zariski decomposition (see Proposition 2.1) is a linear function on each cone in Σ. The fan Σ is determined by the "shadow" of the cone $\mathbf{E}^+$ from $E_1, \ldots, E_r$. The abstract complex of polyhedral sets $\mathcal{P}$ is defined by refining Σ in such a way that we have good properties for the divisor $\overline{D}_{\underline{n}}$ (called the Laufer divisor in [7]) associated to $D_{\underline{n}}$, which is determined by the properties that $D_{\underline{n}} \leq \overline{D}_{\underline{n}}$, $\overline{D}_{\underline{n}} \in \mathbf{E}^+$ and $H^0(X, \mathcal{O}_X(-\overline{D}_{\underline{n}})) = H^0(X, \mathcal{O}_X(-D_{\underline{n}}))$. Then, the essential contribution to $\varphi(\underline{n})$ is $h^1(X, \mathcal{O}_X(-\overline{D}_{\underline{n}}))$.

In the case where $\text{spec}(R)$ is a rational singularity, we have that $H^1(X, \mathcal{O}_X(-\overline{D}_{\underline{n}}))$ $= 0$ for all $\underline{n}$ [19], so that $\varphi(\underline{n}) = 0$ is a periodic function (on the abstract complex $\mathcal{P}$), and we conclude that the series g is rational.

Therefore, we are reduced to understanding the function $h^1(X, \mathcal{O}_X(-\overline{D}_{\underline{n}})$.

When $r = 2$ and $k = R/m$ has characteristic zero, this function has a very nice form. $h^1(X, \mathcal{O}_X(-\overline{D}_{\underline{n}}))$ is in fact semi-periodic (Definition 6.5) on an abstract complex of polyhedral sets, and the series g is thus rational. This is the case $r = 2$ of Theorem 6.6 and Theorem 7.7. The proof is a generalization of Theorem 9 in [9].

An example showing a nontrivial (but semi-periodic) function $h^1(X, \mathcal{O}_X(-\overline{D}_{\underline{n}}))$ is given in section 10.

When $r = 2$ and R contains a field k of characteristic $p > 0$, we give an example where $h^1(X, \mathcal{O}_X(-\overline{D}_{\underline{n}})$ is not semi-periodic and g is not rational in section 8. This is an extension of example 5 in [9].

In section 9 we give an example of $X \to \mathrm{spec}(R)$ such that $r = 3$ and $\mathbf{k} = R/m$ has characteristic 0, $h^1(X, \mathcal{O}_X(-\overline{D}_{\underline{n}}))$ is not semi-periodic and g is not rational. This example is obtained by first constructing a resolution $X_1 \to \mathrm{spec}(R)$ where X_1 has only a single irreducible exceptional divisor ($r = 1$), and then blowing up 2 points on X_1 to construct $X \to \mathrm{spec}(R)$. Thus the function h is semi-periodic on X_1, and the resulting Poincaré series g on X_1 is rational. In particular, we see that the rationality of g depends on the resolution of $\mathrm{spec}(R)$.

In Theorem 6.6 and Theorem 7.7 we prove that if $\mathbf{k} = R/m$ is algebraically closed of characteristic zero, and the group of numerically trivial line bundles $\mathrm{Pic}^0(X)$ on X is a semi-abelian variety, then $h^1(X, \mathcal{O}_X(-\overline{D}_{\underline{n}}))$ is semi-periodic (in an abstract complex of polyhedral sets) and g is rational. A semi-abelian variety is an extension of an abelian variety by a product of multiplicitive groups $\mathbf{G}_m$. $\mathrm{Pic}^0(X)$ is semi-abelian if and only if the divisor class group of $\mathrm{Cl}(R)$ is an extension of a finite group by a semi-abelian variety, since $\mathrm{Cl}(R)$ is an extension of a finite group by $\mathrm{Pic}^0(X)$ (cf. [19]). Hence the condition $\mathrm{Pic}^0(X)$ semi-abelian only depends on R. If the reduced exceptional locus of X is a simple normal crossings divisor E, then $\mathrm{Pic}^0(X)$ is a semi-abelian variety if and only if $\mathrm{Pic}^0(X) \cong \mathrm{Pic}^0(E)$ (cf. Proposition 5.6). Thus if $\mathcal{L}$ is a line bundle of degree $> 2g - 2$ on a nonsingular curve C of genus g, then the (completion of the) contraction of the zero section of $\mathrm{Proj}(\mathcal{O}_C \oplus \mathcal{L})$ is a singularity with class group which is an extension of a finite group by a semi-abelian variety.

To prove Theorem 6.6 we make use of Lang's conjecture (proven by McQuillan [21]), which states that if H is a finitely generated subgroup of a semi-abelian variety G and Y is an irreducible subvariety of G such that $Y \cap H$ is Zariski dense in Y, then Y is a translation of a semi-abelian subvariety of G. Theorem 8 of [9] (which generalizes to prove the $r = 2$ case of Theorem 6.6) uses a very general form of Lang's conjecture for cyclic subgroups of a characteristic 0 algebraic group. This is proven in Theorem 7 of [9].

In Theorem 3.1 we show that topological information on the singularity can be extracted from the series g. In fact, the intersection matrix of the exceptional curves of the resolution $X \to \mathrm{spec}(R)$ as well as $h^1(X, \mathcal{O}_X)$ can be extracted from g. This result could be compared with the result that the Alexander polynomial of a curve singularity can be recovered from a corresponding series [5]. A related series is shown to be rational for rational surface singularities ($H^1(X, \mathcal{O}_X) = 0$) in [6].

An interesting remaining question when $\mathbf{k} = R/m$ algebraically closed of characteristic zero is if there is a good necessary and sufficient condition on $\mathrm{Cl}(R)$ for the existence of a resolution $X \to \mathrm{spec}(R)$ such that the series g is irrational. This question can be compared with the characterizations of surface singularities with torsion divisor class group in [8].

If $\mathbf{k}$ is a field, and $f \in \mathbf{k}[[t_1, \ldots, t_r]]$ is a formal power series in the variables $t_1, \ldots, t_r$, we will say that f is rational if there exist polynomials $P, Q \in \mathbf{k}[t_1, \ldots, t_r]$ such that $f = \frac{P}{Q}$.

2. The Riemann-Roch formula for high multiples of a divisor on the resolution of a surface singularity

This section is a summary of some of the results of section 8 in [9]. Suppose that R is an excellent, normal local ring of dimension 2, with maximal ideal m. Let

$$f : X \to \operatorname{spec}(R)$$

be a resolution of singularities, with integral exceptional divisors $E_1, \ldots, E_r$.

If $\mathcal{L}$ is a line bundle (or a divisor D) on X and C is an integral curve on X, then $\mathcal{L} \cdot C$ (or $D \cdot C$) will denote the line bundle $\mathcal{L} \otimes \mathcal{O}_C$ (or the linear equivalence class of $\mathcal{O}_C \otimes \mathcal{O}_X(D)$). $(\mathcal{L} \cdot C)$ (or $(C \cdot D)$) will denote the degree of $\mathcal{L} \otimes \mathcal{O}_C$ (or of $\mathcal{O}_X(D) \otimes \mathcal{O}_C$).

If $\mathcal{M}$ is a coherent sheaf on X, then $H^1(X, \mathcal{M})$ has finite length as an R-module. We will denote

$$h^1(X, \mathcal{M}) = \ell(H^1(X, \mathcal{M})).$$

The intersection matrix $(E_i \cdot E_j)_{1 \le i,j \le r}$ is negative definite (section 1 of [23], Lemma 14.1 of [19]).

Let us consider the lattice $\mathbf{E} := \bigoplus_{i=1}^r \mathbf{Z} E_i$ and the semigroup

$$\mathbf{E}^+ := \{D \in \mathbf{E} \mid \mathcal{O}_X(-D) \text{ is nef, i.e. } (D \cdot E_i) \le 0 \text{ for } 1 \le i \le r\}. \tag{3}$$

Let $\mathbf{E_Q} := \bigoplus_{i=1}^r \mathbf{Q} E_i$, and let $\mathbf{E}_\mathbf{Q}^+$ be the rational convex polyhedral cone in $\mathbf{E_Q}$ generated by $\mathbf{E}^+$, i.e. $\mathbf{E}_\mathbf{Q}^+ = \bigoplus_{D \in \mathbf{E}^+} \mathbf{Q}_{\ge 0} D$. Then $\mathbf{E}_\mathbf{Q}^+$ is a cone contained in $\bigoplus_{i=1}^r \mathbf{Q}_{\ge 0} E_i$ ([19], p. 238). Therefore, it is a strongly convex cone, i.e. $\mathbf{E}_\mathbf{Q}^+ \cap (-\mathbf{E}_\mathbf{Q}^+) = \{0\}$.

Let $Y = \operatorname{spec}(R)$. Suppose that $D = \sum a_i E_i$ is a divisor with exceptional support. Then

$$H^0(X, \mathcal{O}_X(-D)) \subset H^0(\operatorname{spec}(R) - \{m\}, \mathcal{O}_Y) = R$$

(since R is normal of dimension 2) is an m-primary ideal. In fact,

$$H^0(X, \mathcal{O}_X(-D)) = H^0(X, \mathcal{O}_X(-\sum b_i E_i)),$$

where $b_i = \max\{0, a_i\}$.

Proposition 2.1 ([30], Theorem 7.7)**.** *There exists a unique effective* $\mathbf{Q}$*-divisor* $B = \sum b_i E_i$ *such that*

(i) $\Delta = D + B$ *is in* $\mathbf{E}_\mathbf{Q}^+$*, that is* $(\Delta \cdot E_i) \le 0$ *for* $1 \le i \le r$*, and*
(ii) $(\Delta \cdot E_i) = 0$ *if* E_i *is a component of* B*.*

We will call Δ the *Zariski* $\mathbf{Q}$*-divisor associated to* D.

Proposition 2.2 ([7], Proposition 1)**.** *Among the divisors* $D' \in \mathbf{E}^+$ *such that* $D' \ge D$ *there is a minimal one* $\overline{D}$*. It can be computed by applying the following algorithm: Let* $\hat{D}_1 := D$ *and, for* $i \ge 1$*, let* $\overline{D} := \hat{D}_i$ *if* $\hat{D}_i \in \mathbf{E}^+$*, or else* $\hat{D}_{i+1} := \hat{D}_i + E_{j_i}$ *where* E_{j_i} *is such that* $(\hat{D}_i \cdot E_{j_i}) > 0$*.*

We will call $\overline{D}$ the *Laufer divisor associated to* D, since the previous algorithm is a generalization of Laufer's construction of the fundamental cycle ([17], Prop. 4.1). Note that $\overline{D}$ is the unique divisor in $\mathbf{E}^+$ such that

$$H^0(X, \mathcal{O}_X(-D)) = H^0(X, \mathcal{O}_X(-\overline{D})).$$

Given two $\mathbf{Q}$-divisors D_1, D_2, let us write $D_1 \leq D_2$ if $D_2 - D_1$ is effective.

Lemma 2.3. *The following hold:*

(i) $D \leq \Delta \leq \overline{D}$.

(ii) *For* $n \in \mathbf{N}$, $n\Delta$ *is the Zariski* $\mathbf{Q}$*-divisor associated to* nD.

(iii) *Choose an integer* s *such that* $s\Delta$ *is an integral divisor. Suppose that* n *is a natural number, and* $n = as + b$ *with* $0 \leq b < s$. *Then the natural inclusion*

$$\mathcal{O}_X(-as\Delta - bD) \to \mathcal{O}_X(-nD)$$

induces an isomorphism of global sections

$$H^0(X, \mathcal{O}_X(-as\Delta - bD)) \cong H^0(X, \mathcal{O}_X(-nD)).$$

Proof. The first inequality in (i) holds since the $\mathbf{Q}$-divisor B in Proposition 2.1 is effective. The second one follows from [30], Corollary 7.2, since $\overline{D} - D$ is effective and, for any $E_i \in \operatorname{Supp} B$,

$$\big(((\overline{D} - D) - B) \cdot E_i\big) = (\overline{D} \cdot E_i) \leq 0.$$

For (ii), note that nB satisfies (i) and (ii) in Proposition 2.1 for nD.

For (iii), we have $nD \leq$ as $\Delta + bD \leq n\Delta \leq \overline{nD}$. Hence

$$H^0(X, \mathcal{O}_X(-\overline{nD})) \subseteq H^0(X, \mathcal{O}_X(-as\Delta - bD)) \subseteq H^0(X, \mathcal{O}_X(-nD)).$$

Since $H^0(X, \mathcal{O}_X(-nD)) = H^0(X, \mathcal{O}_X(-\overline{nD}))$, we conclude (iii). $\square$

Lemma 2.4. *Suppose that* $m_1, \ldots, m_r \in \mathbf{Z}$. *There exists a constant* c *such that if* $\mathcal{L}$ *is a line bundle on* X *with* $(\mathcal{L} \cdot E_i) \geq m_i$ *for* $1 \leq i \leq r$, *then*

$$h^1(X, \mathcal{L}) \leq c.$$

Proof. Let A be an effective divisor on X with exceptional support such that $-A$ is ample. There exist $s > 0$, a function

$$\sigma : \{1, \ldots, s\} \to \{1, \ldots, r\},$$

and a sequence of divisors F_i, $1 \leq i \leq s$, such that $F_0 = 0$, $F_i = F_{i-1} + E_{\sigma(i)}$ for $1 \leq i \leq s$, and $A = F_s$. We have exact sequences

$$0 \to \mathcal{O}_{E_{\sigma(i+1)}}(-F_i) \to \mathcal{O}_{F_{i+1}} \to \mathcal{O}_{F_i} \to 0 \tag{4}$$

for $1 \leq i \leq s - 1$.

Each E_i is a locally complete intersection, so by the Riemann-Roch theorem (cf. Exercise IV 1.9 in [14]) $H^1(X, \mathcal{M} \otimes \mathcal{O}_{E_i})) = 0$ if $\mathcal{M}$ is a line bundle on X such that

$$(\mathcal{M} \cdot E_i) > 2p_a(E_i) - 2.$$

Thus if $\mathcal{F}$ is a line bundle on E_i, we have that

$$h^0(E_i, \mathcal{F}) = 0 \text{ if } \deg \mathcal{F} < 0,$$

and

$$h^0(E_i, \mathcal{F}) = \deg \mathcal{F} + 1 - p_a(E_i) \text{ if } \deg \mathcal{F} > 2p_a(E_i) - 2.$$

We get the further approximation

$$h^0(E_i,\mathcal{F}) \le p_a(E_i) \text{ if } 0 \le \deg \mathcal{F} \le 2p_1(E_i)-2.$$

This can be seen as follows. Let

$$t = 2p_a(E_i) - 1 - \deg \mathcal{F} > 0,$$

and let $p_1,\dots,p_t$ be nonsingular points on E_i. Let $\mathcal{G} = \mathcal{O}_{E_i}(p_1+\cdots+p_t)$, a line bundle on E_i of degree t with a nonvanishing section. Thus we have an inclusion

$$H^0(E_i,\mathcal{F}) \to H^0(E_i,\mathcal{F}\otimes\mathcal{G}),$$

and

$$h^0(E_i,\mathcal{F}) \le h^0(E_i,\mathcal{F}\otimes\mathcal{G}) = (2p_1(E_i)-1)+1-p_a(E_i) = p_a(E_i).$$

We have exact sequences

$$0 \to \mathcal{O}_A(-nA) \to \mathcal{O}_{(n+1)A} \to \mathcal{O}_{nA} \to 0 \tag{5}$$

for all $n>0$. By (4) there exists n_0 such that $n \ge n_0$ implies

$$H^1(X,\mathcal{O}_A(-nA)\otimes\mathcal{L}) = 0$$

for any line bundle $\mathcal{L}$ on X such that $(\mathcal{L}\cdot E_i) \ge m_i$ for $1 \le i \le r$. By the formal function theorem,

$$H^1(X,\mathcal{L}) = \varprojlim H^1(X,\mathcal{L}\otimes\mathcal{O}_{nA}) = H^1(X,\mathcal{L}\otimes\mathcal{O}_{n_0A}).$$

Let ω_{E_i} be a canonical divisor on E_i. Set

$$\begin{aligned} d = \max\{&p_a(E_{\sigma(1)}), \deg(\omega_{E_{\sigma(1)}}\otimes\mathcal{O}_X(F_0+(n_0-1)A)) - m_{\sigma(1)} + 1 - p_a(E_{\sigma(1)}),\dots,\\ &p_a(E_{\sigma(s)}), \deg(\omega_{E_{\sigma(s)}}\otimes\mathcal{O}_X(F_{s-1}+(n_0-1)A)) - m_{\sigma(s)} + 1 - p_a(E_{\sigma(s)})\}. \end{aligned}$$

$$\begin{aligned} h^1(X,\mathcal{L}) &= h^1(X,\mathcal{L}\otimes\mathcal{O}_{n_0A})\\ &\le \sum_{j=0}^{n_0-1} h^1(X,\mathcal{O}_A(-jA)\otimes\mathcal{L})\\ &\le \sum_{0\le j\le n_0-1}\left(\sum_{1\le i\le s} h^1(X,\mathcal{O}_{E_{\sigma(i)}}(-F_{i-1}-jA)\otimes\mathcal{L})\right)\\ &= \sum_{0\le j\le n_0-1}\left(\sum_{1\le i\le s} h^0(X,\mathcal{O}_X(F_{i-1}+jA)\otimes\mathcal{L}^{-1}\otimes\omega_{E_{\sigma(i)}})\right)\\ &\le n_0sd. \end{aligned}$$ □

Lemma 2.5. *For fixed b, $0 \le b < s$, the function*

$$\sigma_b(a) = h^1(X,\mathcal{O}_X(-as\Delta - bD))$$

is bounded.

Proof. This is immediate from Lemma 2.4. □

We recall the "local" Riemann-Roch theorem, proved in Lemma 23.1 of [19] as well as in [15]:

(6)
$$\ell(R/H^0(X,\mathcal{O}_X(-D)) = -\frac{1}{2}\left((K_X\cdot D)+(D^2)\right)+h^1(X,\mathcal{O}_X)-h^1(X,\mathcal{O}_X(-D)),$$

where K_X is a canonical divisor of X. For $n = as + b$, $0 \le b < s$ and $a \ge 0$, let us replace D by $as\Delta + bD$ in (6), apply Lemma 2.3 (iii), take $as = n - b$, and use the identity $(\Delta \cdot D) = (\Delta^2)$, which follows since all components of $\Delta - D$ have intersection number 0 with Δ. Then we have

(7)
$$\begin{aligned}\ell\left(R/H^0(X,\mathcal{O}_X(-nD))\right) = &-\frac{1}{2}(\Delta^2)n^2-\frac{1}{2}(\Delta\cdot K_X)n\\ &+\frac{b^2}{2}(\Delta^2)+\frac{b}{2}(\Delta\cdot K_X)-\frac{b^2}{2}(D^2)-\frac{b}{2}(D\cdot K_X)\\ &+h^1(X,\mathcal{O}_X)-\sigma_b(a).\end{aligned}$$

At this point we have obtained the following "local" form of a theorem ((1) of "A summary of principal results" [30]) of Zariski for projective surfaces.

Proposition 2.6. *Let R be an excellent, 2-dimensional equicharacteristic normal local ring. Let $f : X \to \mathrm{spec}(R)$ be a resolution of singularities and let $D \ne 0$ be a divisor on X supported on the exceptional divisor. Then there exist a natural number m, quadratic polynomials $Q_i(n)$ for $1 \le i \le m$, and a function $\gamma : \mathbf{N} \to \{1,\dots,m\}$ such that*

$$\ell\left(R/H^0(X,\mathcal{O}_X(-nD)\right) = Q_{\gamma(n)}(n)$$

for $n \in \mathbf{N}$.

A slightly different proof of this proposition appears in [22].

An extremely interesting question is if the function σ is eventually periodic. This is the "local" form of the question raised by Zariski ((2) of "a summary of principal results" [30]) for projective surfaces. This question of Zariski (and the corresponding local question above) is completely solved in [9]. We prove ([9], Theorem 9) that the answer to the question is "true" in characteristic 0 or over a finite field, but give examples ([9], Example 5) to show that it is "false" in general in positive characteristic.

Suppose that $-F$ is an ample divisor with exceptional support on X. For all $m \ge 0$, there is an exact sequence

$$0 \to \mathcal{O}_F(-mF) \to \mathcal{O}_{(m+1)F} \to \mathcal{O}_{mF} \to 0. \tag{8}$$

There exist constants c_i, $1 \le i \le r$, such that if D' is a divisor on X with $(D' \cdot E_i) \ge c_i$ for $1 \le i \le r$, then $H^1(F, \mathcal{O}_F(D')) = 0$. This follows from the proof of Lemma 2.4.

Lemma 2.7. (i) *There exists a natural number m_0 such that $m \ge m_0$ implies that the natural restriction map*

$$H^1(X,\mathcal{O}_X(-as\Delta-bD)) \to H^1(\mathcal{O}_{mF}(-as\Delta-bD))$$

is an isomorphism for $0 \le b < s$ and $a \ge 0$.

(ii) *There exists an effective divisor C on X with exceptional support for f such that $\mathcal{O}_C(-\Delta)$ is numerically trivial and the restriction map*

$$H^1(X, \mathcal{O}_X(-as\Delta - bD)) \to H^1(C, \mathcal{O}_C(-as\Delta - bD))$$

is an isomorphism for $0 \le b < s$ and $a >> 0$.

Proof. Since $-\Delta$ is nef and $-F$ is ample, there exists m_0 such that $m \ge m_0$ implies

$$((-mF - as\Delta - bD) \cdot E_i) \ge c_i$$

for $0 \le b < s$ and all $a \ge 0$. From the exact sequence (8) we see that

$$H^1(\mathcal{O}_{(m+1)F}(-as\Delta - bD)) \cong H^1(\mathcal{O}_{mF}(-as\Delta - bD))$$

for $m \ge m_0$. By the formal implicit function theorem, (i) follows.

Let $m_0 F = C + C'$, where

$$\text{Supp } C = \{E_i \mid \Delta \cdot E_i = 0\} \quad \text{and} \quad \text{Supp } C' = \{E_i \mid (\Delta \cdot E_i) < 0\}.$$

For $a >> 0$ we have $H^1(X, \mathcal{O}_{C'}(-C - as\Delta - bD)) = 0$; hence the restriction map

$$H^1(X, \mathcal{O}_{m_0F}(-as\Delta - bD)) \to H^1(X, \mathcal{O}_C(-as\Delta - bD))$$

is an isomorphism, and (ii) holds. □

Theorem 2.8 ([9]). *Suppose that R is equicharacteristic and that R has residue characteristic 0, or R has a finite residue field. Then the functions*

$$\sigma_b(a) = h^1(X, \mathcal{O}_X(-as\Delta - bD))$$

are periodic for $a >> 0$.

Proof. When the residue field is finite, it follows as in Theorem 3 of [9]. If R has equicharacteristic 0, it follows from Theorem 7 of [9] applied to each of the connected components of the Zariski closure in $\text{Pic}^0(C)$ of the cyclic group generated by the class of $\mathcal{O}_C(-s\Delta)$, where C is the curve of Lemma 2.7 (ii). Theorem 7 in [9] is stated as follows: "Let G be a connected commutative algebraic group defined over an algebraically closed field $\mathbf{k}$ of characteristic 0, let $x \in G(k)$ and suppose that the cyclic group $\langle x \rangle$ is Zariski dense in G. Then, for any subvariety $\Omega \subset G$, $\Omega \ne G$, the set $\Omega \cap \langle x \rangle$ is finite". □

As a consequence of Theorem 2.8 and (7), we have

Theorem 2.9 ([9], Theorem 9). *Let R be an excellent, 2-dimensional equicharacteristic normal local ring with a residue field that is either of characteristic zero or a finite field. Let $f : X \to \text{spec}(R)$ be a resolution of singularities and let $D \ne 0$ be a divisor on X supported on the exceptional divisor. Then there are a quadratic polynomial $Q(n)$ with coefficients in $\mathbf{Q}$ and a periodic function $\lambda : \mathbf{N} \to \mathbf{Q}$ such that*

$$\ell\left(R/H^0(X, \mathcal{O}_X(-nD)\right) = Q(n) + \lambda(n). \tag{9}$$

for all n sufficiently large.

Therefore,

$$\sum_{n \in \mathbf{N}} \ell\left(R/H^0(X, \mathcal{O}_X(-nD)\right) t^n$$

is a rational series, i.e. a quotient of two polynomials.

3. A Hilbert function of R

Let notation be as in the previous section. For $\underline{n} := (n_1, \dots, n_r) \in \mathbf{Z}^r$, set $D_{\underline{n}} := \sum_{i=1}^r n_i E_i$, and let

$$R^* := \bigoplus_{\underline{n} \in \mathbf{N}^r} H^0(X, \mathcal{O}_X(-D_{\underline{n}})).$$

For $\underline{n} \in \mathbf{N}^r$, let

$$h(\underline{n}) := \ell\left(R/H^0(X, \mathcal{O}_X(-D_{\underline{n}}))\right). \tag{10}$$

h can be viewed as a Hilbert-Samuel function of the generally non-Noetherian ring R^*. It contains a lot of information about the resolution $f : X \to \operatorname{spec}(R)$, as we will show.

Given $\underline{m} = (m_1, \dots, m_r) \in \mathbf{Z}^r$, let $Y = \operatorname{spec}(R)$, $I_{\underline{m}} = H^0(X, \mathcal{O}_X(-D_{\underline{m}}))$. Then

$$I_{\underline{m}} \subset H^0(Y - \{m\}, \mathcal{O}_Y) = R,$$

since R is normal of dimension 2. Also,

$$\begin{aligned} I_{\underline{m}} = H^0(X, \mathcal{O}_X(-D_{\underline{m}})) &= \{f \in R \mid \nu_i(f) \geq m_i,\ 1 \leq i \leq r\} \\ &= \{f \in R \mid \nu_i(f) \geq \max(0, m_i),\ 1 \leq i \leq r\}. \end{aligned}$$

We will prove the following theorem in this section.

Theorem 3.1. *From the function $h(\underline{n})$ we can recover the intersection matrix $(E_i \cdot E_j)_{1 \leq i,j \leq r}$ and the arithmetic genus $h^1(X, \mathcal{O}_X)$ of X and of the E_i, $1 \leq i \leq r$.*

Remark. If R has a rational surface singularity, then the converse to Theorem 3.1 is true, i.e. the intersection matrix $(E_i \cdot E_j)_{1 \leq i,j \leq r}$ is data equivalent to the function $h(\underline{n})$. In fact, knowing the intersection matrix, for any $\underline{n} \in \mathbf{N}^r$, we can determine the Laufer divisor $\overline{D_{\underline{n}}}$ associated to $D_{\underline{n}}$. Then, $H^1(X, \mathcal{O}_X(-\overline{D_{\underline{n}}})) = 0$; thus

$$h(\underline{n}) = -\frac{1}{2}\left((\overline{D_{\underline{n}}} \cdot K_X) + (\overline{D_{\underline{n}}}^2)\right)$$

by (3), and $(\overline{D}_{\underline{n}} \cdot K_X)$ is known since $p_a(E_i) = 0$ for all i.

We now prove Theorem 3.1.

For $1 \leq i \leq r$, let $R_i \subset \mathbf{E}^+_{\mathbf{Q}}$ be the ray of solutions v to the equations

$$(v \cdot E_j) = 0 \text{ for } i \neq j$$

and

$$(v \cdot E_i) \leq 0.$$

Since $(E_i \cdot E_j)$ is negative definite, if $v \in R_i$, then v is an effective $\mathbf{Q}$-divisor,

$$\mathbf{E}^+_{\mathbf{Q}} = R_1 + \cdots + R_r. \tag{11}$$

Let $e_i \in \mathbf{Z}^r$ be the vector which is 1 in the ith coefficent and 0 in all other coefficients.

Lemma 3.2. *Suppose that $\underline{n} \in \mathbf{N}^r$. $D_{\underline{n}} \in \mathbf{E}$ is such that $-D_{\underline{n}}$ not nef if and only if there exist $s \in \mathbf{N}$ and $i \in \{1, \dots, r\}$ such that*

$$h(m(s\underline{n} + e_i)) = h(ms\underline{n}) \tag{12}$$

for all $m > 0$.

Proof. Let B_n be the effective divisor which is the fixed component of the linear system associated to $H^0(X, \mathcal{O}_X(-nD))$. If $D \in \mathbf{E}$ is such that $-D$ is nef, then the set of divisors B_n is bounded from above (as in Theorem 10.1 of [30]). Thus the condition (12) cannot occur. If $\underline{n} \in \mathbf{N}^r$ and $-D_{\underline{n}}$ is not nef, then there is a Zariski decomposition $\Delta = D_{\underline{n}} + B$ with $B \neq 0$. Choose $s \in \mathbf{N}$ so that $s\Delta$ is integral, and i such that E_i is in the support of B. (12) now follows from Lemma 2.3. □

Lemma 3.3. *The rays R_i of* (11) *are determined by the function h.*

Proof. The anti-nef divisors $\mathbf{E}^+$ in $\mathbf{E_Q}$ can be determined from h by Lemma 3.2. The cone $\mathbf{E}^+_{\mathbf{Q}}$ is thus determined by the function h, and thus the set of edges $S = \{R_1, \dots, R_r\}$ of $\mathbf{E}^+_{\mathbf{Q}}$ is determined by h. The individual edges R_i can now be distinguished by h, by Lemma 3.2 and the fact that if $0 \neq \Delta$ is in an edge of $\mathbf{E}^+_{\mathbf{Q}}$, then Δ is in a particular ray R_i precisely when there exists $\epsilon > 0$ such that $-\Delta + \epsilon E_i$ is nef. □

Choose $0 \neq \Lambda_i \in R_i$. There exist (known) $a_{ij} \in \mathbf{Q}$ such that

$$\Lambda_i = \sum a_{ij} E_j \tag{13}$$

for $1 \leq i \leq r$. Let $(b_{ij}) = (a_{ij})^{-1}$. We then have

$$E_i = \sum b_{ij} \Lambda_j. \tag{14}$$

From (7) and Lemma 2.5 we can determine $(\Lambda_j \cdot K_X)$ for $1 \leq j \leq r$. Then from (14) we can calculate $(E_i \cdot K_X)$ for $1 \leq i \leq r$.

Now choose integral divisors (with known $c_{kj} > 0$)

$$H_k = -(c_{k1}E_1 + \cdots + c_{kr}E_r)$$

for $1 \leq k \leq r$ so that the H_k are linearly independent in $\mathbf{E_Q}$ and the $-H_k$ lie in the interior of $\mathbf{E}^+$. Let $\underline{c}_k = (c_{k1}, \dots, c_{kr}) \in \mathbf{N}^r$. Each H_k is ample, since $(E_i \cdot H_k) > 0$ for all i. Thus there exists $m_0 > 0$ such that

$$h^1(X, \mathcal{O}_X(mH_k)) = h^1(X, \mathcal{O}_X(mH_k + E_i)) = 0$$

for $m \geq m_0$ and $1 \leq i, k \leq r$.

By (6), $m \geq m_0$ implies $h(m\underline{c}_k)$, $h(m\underline{c}_k - e_i)$ are quadratic polynomials in m for $1 \leq i, k \leq r$. Thus

$$h(m\underline{c}_k) = \frac{1}{2}\left((mH_k \cdot K_X) - (mH_k^2)\right) + \ell(H^1(X, \mathcal{O}_X))$$

for $m \geq m_0$, and we can determine $(H_k \cdot K_X)$:

$$\begin{aligned} h(m\underline{c}_k - e_i) &= -\tfrac{1}{2}(H_k^2)m^2 + \left((H_k \cdot (\tfrac{1}{2}K_X - E_i)\right)m \\ &\quad + \tfrac{1}{2}\left(E_i \cdot (K_X - E_i)\right) + \ell(H^1(X, \mathcal{O}_X)) \end{aligned}$$

for $m \geq m_0$. We can now determine $(H_k \cdot E_i)$ for $1 \leq i, k \leq r$. We can thus determine the matrix

$$(E_k \cdot E_i) = -(c_{kj})^{-1}(H_j \cdot E_i).$$

Now we can recover the arithmetic genus $p_a(E_i)$ of each E_i from the formula

$$2p_a(E_i) - 2 = (E_i^2) + (E_i \cdot K_X).$$

Since $h^1(X, \mathcal{O}_X)$ is known, $p_a(X)$ is also recovered from h, and hence we have proved Theorem 3.1.

In this paper we will study the Hilbert-Samuel function $h(\underline{n})$. We will consider the question of giving a result analogous to formula (9) of Theorem 2.9 for the function h. See Theorem 6.6 for a precise statement generalizing these results. In particular, we will study when the series

$$\sum_{\underline{n}\in\mathbf{N}^r} h(\underline{n})t^{\underline{n}}$$

is rational.

4. Piecewise linearity of the Zariski Q-divisor and piecewise periodicity of its difference with the Laufer divisor

Suppose that R is a complete normal local ring of dimension two, and $f : X \to \operatorname{spec}(R)$ is a resolution of singularities with integral exceptional divisors $E_1, \ldots, E_r$. Let $\mathbf{E}^+$ be the semigroup in (3) and $\mathbf{E}^+_{\mathbf{Q}}$ the associated cone, which is contained in $\bigoplus_{i=1}^r \mathbf{Q}_{\geq 0}E_i$. For $1 \leq i \leq r$, let $\Delta_i \in \mathbf{E}_{\mathbf{Q}} = \bigoplus_{i=1}^r \mathbf{Q}E_i$ be defined by the condition $(\Delta_i \cdot E_j) = -\delta_{ij}$, so that $\mathbf{E}^+_{\mathbf{Q}} = \bigoplus_{i=1}^r \mathbf{Q}_{\geq 0}\Delta_i$.

Given $\underline{n} \in \mathbf{Q}^r$, let $D_{\underline{n}} = \sum_{i=1}^r n_iE_i \in \mathbf{E}_{\mathbf{Q}}$. For $\underline{n} \in \mathbf{N}^r$, let $\Delta_{\underline{n}}$ (resp. $\overline{D_{\underline{n}}}$) be the Zariski $\mathbf{Q}$-divisor (resp. Laufer divisor) associated to $D_{\underline{n}}$.

Definition 4.1. Let S be a subset of $\{1, \ldots , r\}$. For $\underline{n} \in \mathbf{Q}^r_{\geq 0}$, let $\Delta^S_{\underline{n}}$ be the orthogonal projection of $D_{\underline{n}}$ on $\{E_i\}^{\perp}_{i\in S}$, i.e. $\Delta^S_{\underline{n}} \in \bigoplus_{i=1}^r \mathbf{Q}E_i$ is defined by

$$\text{Supp }(\Delta^S_{\underline{n}} - D_{\underline{n}}) \subseteq \bigcup_{i\in S} E_i, \qquad (\Delta^S_{\underline{n}} \cdot E_i) = 0 \ \text{ for } i \in S. \tag{15}$$

We define

$$\sigma_S := \{\underline{n} \in \mathbf{Q}^r_{\geq 0} \mid \Delta^S_{\underline{n}} - D_{\underline{n}} \geq 0, \ \ \Delta^S_{\underline{n}} \in \mathbf{E}^+_{\mathbf{Q}}\}. \tag{16}$$

Let $\mathbf{F}_S$ be the face of $\mathbf{E}^+_{\mathbf{Q}}$ orthogonal to $\{E_i\}_{i\in S}$; that is, $\mathbf{F}_S = \bigoplus_{j\notin S} \mathbf{Q}_{\geq 0}\Delta_j$. Note that

$$\sigma_S = \{\underline{n} \in \mathbf{Q}^r_{\geq 0} \mid \left(D_{\underline{n}} + (\bigoplus_{i\in S} \mathbf{Q}_{\geq 0}E_i)\right) \cap \mathbf{F}_S \neq \emptyset\}.$$

Given $S \subseteq \{1, \ldots , r\}$, let $S = S_1 \cup \ldots \cup S_k$ be the partition of S determined by the connected components of $\bigcup_{i\in S} E_i$, and let

$$e_S = \text{s.c.m. } \{|\det(E_i \cdot E_j)_{i,j\in S_l}|\}_{l=1,\ldots,k} \,. \tag{17}$$

Let $J := \{1, \ldots , r\} \setminus S$.

Theorem 4.2. *The following hold:*

(i) *Given $\underline{n} \in \mathbf{Q}^r_{\geq 0}$, let $\underline{n}_J = (n_j)_{j\in J}$ be its projection. Then,*

$$\Delta^S_{\underline{n}} = \sum_{j\in J} l^S_j(\underline{n}_J)\Delta_j = D_{\underline{n}} + \sum_{i\in S} b^S_i(\underline{n})E_i, \tag{18}$$

where $l_j(\underline{n}_J)$ (resp. $b^S_i(\underline{n})$) are linear functions on $\underline{n}_J$ (resp. $\underline{n}$) with coefficients in $\frac{1}{e_S}\mathbf{Z}$.

(ii) *For $\underline{n} \in \mathbf{N}^r$, $\underline{n} \in \sigma_S$ if and only if $\Delta^S_{\underline{n}}$ is the Zariski $\mathbf{Q}$-divisor $\Delta_{\underline{n}}$ associated to $D_{\underline{n}}$.*

(iii) *For any $S \subset \{1, \ldots , r\}$ (strictly contained), σ_S is a strongly convex rational polyhedral cone of dimension r. The set Σ consisting of all σ_S's and its faces is a fan subdivision of $\mathbf{Q}^r_{\geq 0}$.*

Proof. For $S \subseteq \{1, \dots, r\}$ and for any $\underline{n} \in \mathbf{Q}^r_{\geq 0}$ we have

$$\Delta^S_{\underline{n}} = D_{\underline{n}} + \sum_{i \in S} b^S_i(\underline{n}) E_i,$$

where the $b^S_i(\underline{n})$ are defined by

$$\sum_{i \in S} b^S_i(\underline{n})(E_i \cdot E_{i'}) = -(D_{\underline{n}} \cdot E_{i'}) \quad \text{for all } i' \in S.$$

Therefore the b^S_i are linear functions on $\underline{n}$ with coefficients in $1/e_S\mathbf{Z}$. We may also write

$$\Delta^S_{\underline{n}} = \sum_{j \in J} l^S_j(\underline{n}) \Delta_j,$$

where the $l^S_j(\underline{n})$ are defined by $\Big((\Delta^S_{\underline{n}} - D_{\underline{n}}) \cdot \Delta_{j'}\Big) = 0$ for $j' \in J$, i.e.

$$\sum_{j \in J} l^S_j(\underline{n})(\Delta_j \cdot \Delta_{j'}) = -n_{j'} \quad \text{for all } j' \in J.$$

Hence the l^S_j are linear functions on $\underline{n}_J$ with coefficients in $\mathbf{Q}$. Since $l^S_j(\underline{n}_J) = -(\Delta^S_{\underline{n}} \cdot E_j)$, the coefficients are in $\frac{1}{e_S}\mathbf{Z}$; hence (i) holds

For (ii), note that $\Delta_{\underline{n}} - D_{\underline{n}}$ effective, $\Delta_{\underline{n}} \in \mathbf{E}^+_{\mathbf{Q}}$ and $(\Delta_{\underline{n}} \cdot E_i) = 0$ for $E_i \in \mathrm{Supp}(\Delta_{\underline{n}} - D_{\underline{n}})$ characterize the Zariski $\mathbf{Q}$-divisor $\Delta_{\underline{n}}$ associated to $D_{\underline{n}}$. Hence (ii) follows from (15) and (16).

For (iii), (16) and (18) imply that

$$\sigma_S = \left\{\underline{n} \in \mathbf{Q}^r_{\geq 0} \mid b^S_i(\underline{n}) \geq 0 \text{ for } i \in S, \quad l^S_j(\underline{n}_J) \geq 0 \text{ for } j \in J\right\} \tag{19}$$

Therefore, σ_S is the intersection of a finite number of rational halfspaces, thus a rational convex polyhedral cone. Since it is contained in $\mathbf{Q}^r_{\geq 0}$, it is strongly convex. If $\dim \sigma_S < r$, then σ_S is contained in a hyperplane H. Since the face $\mathbf{F}_S$ of $\mathbf{E}^+_{\mathbf{Q}}$ orthogonal to S is contained in σ_S, H must be the orthogonal of a divisor $D \in \bigoplus_{i \in S} \mathbf{Z} E_i$. For $i \in S$, the projection of $\mathbf{F}_S$ from E_i in $\bigoplus_{j \neq i} \mathbf{Q} E_j$ is contained in σ_S, thus in H. If $\mathbf{F}_S \neq \{0\}$, i.e. S is strictly contained in $\{1, \dots, r\}$, this implies $(D \cdot E_i) = 0$ and, by the negative definiteness of the intersection matrix restricted to S, $D = 0$. Therefore $\dim \sigma_S = r$.

Let us show that

$$\bigcup_{S \subseteq \{1, \dots, r\}} \sigma_S = \mathbf{Q}^r_{\geq 0}.$$

The inclusion $\subseteq$ is clear, and, since the σ_S's are cones, it suffices to prove that $\mathbf{N}^r \subseteq \bigcup_S (\sigma_S \cap \mathbf{N}^r)$. Let $\underline{n} \in \mathbf{N}^r$ and let $S = \{i \mid E_i \subseteq \mathrm{Supp}(\Delta_{\underline{n}} - D_{\underline{n}})\}$. Then $\Delta^S_{\underline{n}} = \Delta_{\underline{n}}$ by (15), and hence $\underline{n} \in \sigma_S$ by (ii).

In order to prove that Σ is a fan, it is enough to show that, for $S, S' \subset \{1, \dots, r\}$, $\sigma_S \cap \sigma_{S'}$ is a face of σ_S. Let $S, S' \subset \{1, \dots, r\}$; then

$$\sigma_S \cap \sigma_{S'} = \{\underline{n} \in \mathbf{Q}^r_{\geq 0} \mid \Delta^S_{\underline{n}} = \Delta^{S'}_{\underline{n}} \geq D_{\underline{n}},\ \Delta^S_{\underline{n}} \in \mathbf{E}^+_{\mathbf{Q}}\}.$$

In fact, from (ii) follows the equality of both members intersected with $\mathbf{N}^r$. Since they are both cones by (i), and contained in $\mathbf{Q}^r_{\geq 0}$, we conclude the equality. Therefore,

$$\begin{aligned} \sigma_S \cap \sigma_{S'} = \sigma_S \cap \{\underline{n} \in \mathbf{Q}^r_{\geq 0} \mid b_i^S(\underline{n}) = 0 \ \text{ for } i \in S \setminus (S' \cap S), \\ l_j^S(\underline{n}_J) = 0 \quad \text{for } j \in S' \setminus (S \cap S')\} \end{aligned} \tag{20}$$

is a face of σ_S by (19). □

Corollary 4.3. *The fan Σ consisting of the cones σ_S in Definition 4.1 and its faces satisfies the following property: For each $\sigma \in \Sigma$, each of the coefficients of the Zariski $\mathbf{Q}$-divisor, that is, of the function*

$$\sigma \cap \mathbf{Z}^r \to \mathbf{E}_{\mathbf{Q}} \cong \mathbf{Q}^r, \qquad \underline{n} \mapsto \Delta_{\underline{n}},$$

is a linear function of $\underline{n}$ with coefficients in $\mathbf{Q}$. Moreover, if $\sigma \subseteq \sigma_S$, then the coefficients are in $\frac{1}{e_S}\mathbf{N}$.

Next we will subdivide the fan Σ in order to have a certain periodicity for the coefficients of the function $\underline{n} \mapsto \overline{D_{\underline{n}}} - \Delta_{\underline{n}}$ (Theorem 4.9). The subdivision we will give consists of rational convex polyhedral sets. By a *rational convex polyhedral set in $\mathbf{Q}^r$*, or more simply a *polyhedral set*, we mean a set of the form

$$P = \{\underline{n} \in \mathbf{Q}^r \mid L_i(\underline{n}) \geq b_i, \quad 1 \leq i \leq m\}, \tag{21}$$

where $m \in \mathbf{N}$ and, for $1 \leq i \leq m$, L_i is an integral linear form on $\mathbf{Q}^r$ and $b_i \in \mathbf{Z}$. We define the *cone associated to P* to be

$$\sigma_P := \{\underline{n} \in \mathbf{Q}^r \mid L_i(\underline{n}) \geq 0, \quad 1 \leq i \leq m\}.$$

A subset Q of P is called a *face* of P if there exist an integral linear form L on $\mathbf{Q}^r$ and $b \in \mathbf{Z}$ such that

$$P \subseteq \{\underline{n} \in \mathbf{Q}^r \mid L(\underline{n}) \geq b\} \quad \text{ and } \quad Q = P \cap \{\underline{n} \in \mathbf{Q}^r \mid L(\underline{n}) = b\}.$$

Definition 4.4. An abstract complex of polyhedral sets in $\mathbf{Q}^r$ is a finite set $\mathcal{P} = \{P_\gamma\}_{\gamma \in \Lambda}$ of polyhedral sets in $\mathbf{Q}^r$ such that P has dimension r for all $P \in \mathcal{P}$. Given a fan Σ in $\mathbf{Q}^r$, an abstract complex of polyhedral sets $\mathcal{P} = \{P_\gamma\}_{\gamma \in \Lambda}$ is a subdivision of Σ with the same associated cones if

(i) $\bigcup_{\gamma \in \Lambda} P_\gamma$ is the support of Σ, and
(ii) for each $P_\gamma \in \mathcal{P}$, there exists $\sigma \in \Sigma$ such that $P_\gamma \subseteq \sigma$.

Definition 4.5. Let S be a subset of $\{1, \dots, r\}$ and $J = \{1, \dots, r\} \setminus S$. For any pair (α, β), where

$$\alpha : T \to \frac{1}{e_S}\mathbf{N} \qquad \beta : J \setminus T \to \frac{1}{e_S}\mathbf{N}$$

are maps defined on some subset T of J and its complement, we define the polyhedral set in $\mathbf{Q}^r$

(22)
$$P_S(\alpha, \beta) := \{\underline{n} \in \sigma_S \mid l_j^S(\underline{n}) = \alpha(j) \ \text{ for } j \in T, \quad l_j^S(\underline{n}) \geq \beta(j) \ \text{ for } j \in J \setminus T\}$$

where $l_j^S(\underline{n}) = l_j^S(\underline{n}_J)$ are the linear functions with coefficients in $\frac{1}{e_S}\mathbf{Z}$ in Theorem 3.2 (i). We will simply denote $P_S(\alpha, \beta)$ by $P(\alpha, \beta)$ if there is no possible confusion on S.

Note that the cone associated to $P(\alpha,\beta)$ is

$$\sigma_{P(\alpha,\beta)} = \{\underline{n} \in \sigma_S \mid l_j^S(\underline{n}) = 0 \quad \text{for } j \in T\} = \sigma_S \cap \sigma_{S\cup T},$$

which is a face of σ_S (see (20)).

Lemma 4.6. *Let S be a subset of $\{1,\dots,r\}$ and $J = \{1,\dots,r\}\setminus S$. Suppose that an assignment*

$$\alpha \mapsto \alpha^c$$

is given to each map $\alpha : T \to \frac{1}{e_S}\mathbf{N}$, defined on some subset T of J, of a map $\alpha^c : J\setminus T \to \frac{1}{e_S}\mathbf{N}$. Then, there exists an abstract complex of polyhedral sets $\mathcal{P}_{\sigma_S}$ subdividing σ_S with the same associated cones such that

$$\text{for all } P \in \mathcal{P}_{\sigma_S} \text{ there exists a map } \alpha \text{ such that } P \cap \mathbf{Z}^r \subseteq P_S(\alpha,\alpha^c). \tag{23}$$

Therefore, if an assignation as before is given for every subset S of $\{1,\dots,r\}$, then there exists an abstract complex of polyhedral sets $\mathcal{P}$ subdividing the fan Σ in Corollary 3.3 with the same associated cones and such that

$$\text{for all } P \in \mathcal{P} \text{ there exist a set } S \text{ and a map } \alpha \text{ such that } P \cap \mathbf{Z}^r \subseteq P_S(\alpha,\alpha^c).$$

Proof. The second assertion follows from the first (see the remark before Definition 4.5). To prove the first one, we start by defining a finite set Λ of pairs (α,β) such that

$$P(\alpha,\beta) \subseteq P(\alpha,\alpha^c)$$

and

$$\sigma_S \cap \mathbf{Z}^r = \bigcup_{(\alpha,\beta)\in\Lambda} \left(P(\alpha,\beta)\cap \mathbf{Z}^r\right), \tag{24}$$

the union being disjoint.

Let $\Lambda_0 = \{\alpha_0\}$, where $\alpha_0 : \emptyset \to \frac{1}{e_S}\mathbf{N}$ is the trivial map, and let $\beta_0 = \alpha_0^c$. Let $1 \le t \le \sharp J$, and suppose we have defined a finite set Λ_{t-1} of maps $\alpha' : T' \to \frac{1}{e_S}\mathbf{N}$, where $T' \subset J$ has cardinal $\le t-1$, and, for each $\alpha' \in \Lambda_{t-1}$, a map $\beta' : J\setminus T' \to \frac{1}{e_S}\mathbf{N}$, $\beta'(j) \ge \alpha^c(j)$ for $j \in J\setminus T'$, so that $P(\alpha',\beta') \subseteq P(\alpha',\alpha'^c)$. Let Λ_t be the union of Λ_{t-1} and all maps $\alpha : T \to \frac{1}{e_S}\mathbf{N}$, where $\sharp T = t$ and there exists $T' \subset T$ with $\sharp T' = t-1$, such that $\alpha' = \alpha|_{T'}$ belongs to Λ_{t-1} and

$$\alpha(j) < \beta'(j) \quad \text{for} \quad j \in T\setminus T'. \tag{25}$$

For any of these maps α, let $\beta : J\setminus T \to \frac{1}{e_S}\mathbf{N}$,

$$\beta(j) = \sup\left\{\alpha^c(j), \{\widetilde{\beta}(j) \mid \widetilde{\alpha} = (\alpha \mid \widetilde{T}) \in \Lambda_{t-1},\quad \widetilde{\alpha} : \widetilde{T} \to \frac{1}{e_S}\mathbf{N},\quad j \notin \widetilde{T}\}\right\} \quad \text{for } j \in J\setminus T \tag{26}$$

Finally, set $\Lambda := \{(\alpha,\beta) \mid \alpha \in \Lambda_{\sharp J}\}$.

Condition (25) implies that Λ_t, and hence Λ, is a finite set. By (26), we have $\beta(j) \ge \alpha^c(j)$ for $j \in J\setminus T$; hence

$$P(\alpha,\beta) \subseteq P(\alpha,\alpha^c).$$

Let us prove (24). First note that, for any $\alpha' \in \Lambda_{t-1}$, $1 \le t \le \sharp J$, we have

$$P(\alpha',0)\cap\mathbf{Z}^r = (P(\alpha',\beta')\cap\mathbf{Z}^r) \cup \left(\bigcup_{\alpha\in\Lambda_t,\alpha|_{T'}=\alpha'} (P(\alpha,0)\cap\mathbf{Z}^r)\right).$$

In fact, if $\underline{n} \in (P(\alpha',0) \cap \mathbf{Z}^r) \setminus (P(\alpha',\beta') \cap \mathbf{Z}^r)$ then, there exists $j \in J \setminus T'$ such that $l_j^S(\underline{n}) < \beta'(j)$, and $l_j^S(\underline{n}) \in \frac{1}{e_S}\mathbf{N}$ by (i) in Theorem 4.2. Let $T = T' \cup \{j\}$ and $\alpha : T \to \frac{1}{e_S}\mathbf{N}$, defined by $\alpha|_{T'} = \alpha'$, $\alpha(j) = -(\Delta_{\underline{n}} \cdot E_j) = l_j^S(\underline{n})$. Then α satisfies (19); thus $\alpha \in \Lambda_t$, and $\underline{n} \in P(\alpha, 0)$. Since $\sigma_S = P(\alpha_0, 0)$ and for $\alpha \in \Lambda_{\sharp J}$ we have $P(\alpha,\beta) = P(\alpha,0)$, we conclude (24).

In order to prove that the union is disjoint, let $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in \Lambda$, $(\alpha_1,\beta_1) \neq (\alpha_2,\beta_2)$, $\alpha_i : T_i \to \frac{1}{e_S}\mathbf{N}$ for $i = 1,2$. If $T_1 = T_2$, then $\alpha_1 \neq \alpha_2$ and $P(\alpha_1,\beta_1) \cap P(\alpha_2,\beta_2) = \emptyset$. Suppose $T_1 \neq T_2$; we may assume that $T_1 \not\subset T_2$ and that

$$\alpha_1 \mid T_1 \cap T_2 = \alpha_2 \mid T_1 \cap T_2.$$

Let

$$T_1 = T^0 \supset T^1 \supset \cdots \supset T^i \supset T^{i+1} \supset \cdots \supset T^{k+1}$$

be such that $\alpha^i := \alpha_1|_{T^i} \in \Lambda_{t_1 - i}$ where $t_1 = \sharp T_1$, $T^i = T^{i+1} \cup \{j_i\}$, $\alpha_1(j_i) = \alpha^i(j_i) < \beta^{i+1}(j_i)$ and $T^{k+1} \subset T_1 \cap T_2$, $T^k \not\subset T_1 \cap T_2$. Then $k \geq 0$, $j_k \notin T_2$, and

$$\alpha_1(j_k) < \beta^{k+1}(j_k).$$

If $P(\alpha_1,\beta_1) \cap P(\alpha_2,\beta_2) \neq \emptyset$, then $\alpha_1(j_k) \geq \beta_2(j_k)$; but, by (26), $\beta_2(j_k) \geq \beta^{k+1}(j_k)$. Therefore $P(\alpha_1,\beta_1) \cap P(\alpha_2,\beta_2) = \emptyset$.

Now, for any $(\alpha,\beta) \in \Lambda$, let us consider the polyhedral set

$$\overline{P}(\alpha,\beta) = \{\underline{n} \in \sigma_S \mid \alpha(j) - \frac{1}{2e_S} \leq l_j^S(\underline{n}) \leq \alpha(j) + \frac{1}{2e_S} \text{ for } j \in T,$$

$$\beta(j) - \frac{1}{2e_S} \leq l_j^S(\underline{n}) \text{ for } j \in J \setminus T\}, \tag{27}$$

whose associated cone is $\sigma_{P(\alpha,\beta)} = \sigma_S \cap \sigma_{S\cup T}$. We have:

(a) $P(\alpha,\beta) \cap \mathbf{Z}^r = \overline{P}(\alpha,\beta) \cap \mathbf{Z}^r$, and
(b) $\sigma_S = \bigcup_{(\alpha,\beta)\in\Lambda} \overline{P}(\alpha,\beta)$.

This implies that the set $\{\overline{P}(\alpha,\beta)\}_{(\alpha,\beta)\in\Lambda}$ is an abstract complex of polyhedral sets $\mathcal{P}_{\sigma_S}$ subdividing σ_S with the same associated cones and satisfying (23). □

For every subset S of $\{1,\ldots,r\}$, let us construct an assignment $\alpha \mapsto \alpha^c$ as in Lemma 4.6 such that the function $\underline{n} \mapsto \overline{D_{\underline{n}}} - \Delta_{\underline{n}}$ has good properties on the sets $P_S(\alpha,\alpha^c)$.

Let us fix S and $T \subsetneq J = \{1,\ldots,r\} \setminus S$. In a similar way as in Proposition 2.2, given a divisor $D = \sum_{i=1}^r n_i E_i$, among the divisors $D' \geq D$ such that $(D' \cdot E_i) \leq 0$ for all $i \in S \cup T$, there is a minimal one. Let us denote it by $\widetilde{D}^{(S\cup T)}$, or $\widetilde{D}$ if there is no possible confusion on $S \cup T$. For $\underline{n} \in \sigma_S$, we have

$$\overline{D_{\underline{n}}} \geq \widetilde{D_{\underline{n}}} \geq \Delta_{\underline{n}}$$

and $\widetilde{D_{\underline{n}}}$ may be computed as follows: Let $\hat{D}_1 := \lceil \Delta_{\underline{n}} \rceil$, where, if $\Delta_{\underline{n}} = \sum_{i=1}^r q_i E_i$, $q_i \in \mathbf{Q}_{\geq 0}$, then $\lceil \Delta_{\underline{n}} \rceil = \sum_{i=1}^r \lceil q_i \rceil E_i$. For $i \geq 1$, let $\widetilde{D} := \hat{D}_i$ if $(\hat{D}_i \cdot E_j) \leq 0$ for all $j \in S \cup T$, or else $\hat{D}_{i+1} := \hat{D}_i + E_{j_i}$, where $j_i \in S \cup T$ is such that $(\hat{D}_i \cdot E_{j_i}) > 0$.

Given a map $\alpha : T \to \frac{1}{e_S}\mathbf{N}$, let $P(\alpha, 0)$ be the polyhedral set defined by (22) for the map β identically 0. Note that $P(\alpha,0) \supseteq P(\alpha,\beta)$ for any other map β.

Lemma 4.7. *For any map* $\alpha : T \to \frac{1}{e_S}\mathbf{N}$*, if* $\underline{n}, \underline{m} \in P(\alpha, 0) \cap \mathbf{Z}^r$ *and* $\lceil \Delta_{\underline{n}} \rceil - \Delta_{\underline{n}} = \lceil \Delta_{\underline{m}} \rceil - \Delta_{\underline{m}}$ *(for example if* $\underline{n} - \underline{m} \in e_S\mathbf{Z}^r$*), then*

$$\widetilde{D_{\underline{n}}} - \Delta_{\underline{n}} = \widetilde{D_{\underline{m}}} - \Delta_{\underline{m}}.$$

Proof. Let $\hat{D}_1 = \lceil \Delta_{\underline{n}} \rceil < \hat{D}_2 < \cdots < \hat{D}_k = \widetilde{D_{\underline{n}}}$ be defined as above. Let us show by induction on i that $\hat{D}_i \leq \widetilde{D_{\underline{m}}} + (\Delta_{\underline{n}} - \Delta_{\underline{m}})$; hence $\widetilde{D_{\underline{n}}} - \Delta_{\underline{n}} \leq \widetilde{D_{\underline{m}}} - \Delta_{\underline{m}}$ and, in an analogous way, we obtain the other inequality.

For $i = 1$, $\hat{D}_1 = \lceil \Delta_{\underline{n}} \rceil = \lceil \Delta_{\underline{m}} \rceil + (\Delta_{\underline{n}} - \Delta_{\underline{m}}) \leq \widetilde{D_{\underline{m}}} + (\Delta_{\underline{n}} - \Delta_{\underline{m}})$. Suppose $\hat{D}_i \leq \widetilde{D_{\underline{m}}} + (\Delta_{\underline{n}} - \Delta_{\underline{m}})$. Let $\hat{D}_{i+1} = \hat{D}_i + E_{j_i}$, where $j_i \in S \cup T$, $(\hat{D}_i \cdot E_{j_i}) > 0$. If $j_i \in S$, then $(\Delta_{\underline{n}} \cdot E_{j_i}) = (\Delta_{\underline{m}} \cdot E_{j_i}) = 0$; and, if $j_i \in T$, then $(\Delta_{\underline{n}} \cdot E_{j_i}) = (\Delta_{\underline{m}} \cdot E_{j_i}) = -\alpha(E_{j_i})$. Therefore

$$\left(\left(\widetilde{D_{\underline{m}}} + (\Delta_{\underline{n}} - \Delta_{\underline{m}}) - \hat{D}_i\right) \cdot E_{j_i}\right) = \left(\widetilde{D_{\underline{m}}} \cdot E_{j_i}\right) - \left(\hat{D}_i \cdot E_{j_i}\right) < 0.$$

Since $\Delta_{\underline{n}} - \Delta_{\underline{m}} \in \mathbf{E}$, this implies that $E_{j_i} \leq \widetilde{D_{\underline{m}}} + (\Delta_{\underline{n}} - \Delta_{\underline{m}}) - \hat{D}_i$, that is, $\hat{D_{i+1}} \leq \widetilde{D_{\underline{m}}} + (\Delta_{\underline{n}} - \Delta_{\underline{m}})$. $\square$

Lemma 4.8. *For any subset* T *of* J *and any map* $\alpha : T \to \frac{1}{e_S}\mathbf{N}$*, there exists a map* $\alpha^c : J \setminus T \to \frac{1}{e_S}\mathbf{N}$ *such that*

$$\overline{D_{\underline{n}}} = \widetilde{D}^{(S\cup T)} \qquad \text{for } \underline{n} \in P(\alpha, \alpha^c) \cap \mathbf{Z}^r.$$

Therefore

$$\overline{D_{\underline{n}}} - \Delta_{\underline{n}} = \overline{D_{\underline{m}}} - \Delta_{\underline{m}} \quad \text{for } \underline{n}, \underline{m} \in P(\alpha, \alpha^c) \cap \mathbf{Z}^r,\ \underline{n} - \underline{m} \in e_S\mathbf{Z}^r.$$

Proof. There exists $\{\underline{n}_1, \cdots, \underline{n}_t\} \subset P(\alpha, 0) \cap \mathbf{Z}^r$ such that

$$P(\alpha, 0) \cap \mathbf{Z}^r \subset \bigcup_{i=1}^{t} (\underline{n}_i + e_S\mathbf{Z}^r).$$

For $j \in J \setminus T$, let

$$\alpha^c(j) = \sup\{0,\ \left((\widetilde{D}_{\underline{n}_i} - \Delta_{\underline{n}_i}) \cdot E_j\right),\ 1 \leq i \leq t\}.$$

By the previous lemma, if $\underline{n} \in P(\alpha, 0) \cap \mathbf{Z}^r$, then there exists i, $1 \leq i \leq t$, such that $\widetilde{D}_{\underline{n}} - \Delta_{\underline{n}} = \widetilde{D}_{\underline{n}_i} - \Delta_{\underline{n}_i}$. Therefore, if $\underline{n} \in P(\alpha, \alpha^c)$ then $\left(\widetilde{D}_{\underline{n}} \cdot E_j\right) \leq 0$ for all $j \in J \setminus T$, and hence $\overline{D_{\underline{n}}} = \widetilde{D}_{\underline{n}}$. The second assertion follows again from Lemma 4.7 $\square$

From Lemmas 4.6, 4.7 and 4.8 we conclude

Theorem 4.9. *There exists an abstract complex of polyhedral sets* $\mathcal{P}$ *subdividing the fan* Σ *in Corollary 4.3, with the same associated cones, such that, for every* $P \in \mathcal{P}$*, if* $P \subseteq \sigma_S$ *then*

$$\overline{D_{\underline{n}}} - \Delta_{\underline{n}} = \overline{D_{\underline{m}}} - \Delta_{\underline{m}} \quad \text{for } \underline{n}, \underline{m} \in P \cap \mathbf{Z}^r,\ \underline{n} - \underline{m} \in e_S\mathbf{Z}^r. \tag{28}$$

Example 4.10. In an A_4-singularity, whose dual graph is

$$\underset{E_1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{E_2}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{E_3}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{E_4}{\bullet}$$

and all self-intersections are -2, let us consider $\sigma_{\{2,4\}}$, which is given by

$$n_2 \leq \frac{n_1+n_3}{2}, \quad n_4 \leq \frac{n_3}{2}, \quad \frac{n_1}{2} \leq n_3 \leq 3n_1.$$

For $\underline{n} \in \sigma_{\{2,4\}} \cap \mathbf{Z}^4$,

$$\Delta_{\underline{n}} = n_1E_1 + \frac{n_1+n_3}{2}E_2 + n_3E_3 + \frac{n_3}{2}E_4.$$

Let $P_0 := \sigma_{\{2,4\}} \cap \{\underline{n} \mid (\Delta_{\underline{n}} \cdot E_3) = 0\}$ and $P_1 = \sigma_{\{2,4\}} \setminus P_0$, which are rational convex polyhedral sets whose associated cones are the 3-dimensional face P_0 of $\sigma_{\{2,4\}}$ and $\sigma_{\{2,4\}}$ respectively. Then, for $\underline{n} \in P_0 \cap \mathbf{Z}^4$,

$$\overline{D}_{\underline{n}} - \Delta_{\underline{n}} = \begin{cases} 0 & \text{if } 2|n_3, \\ \frac{1}{2}E_2 + E_3 + \frac{1}{2}E_4 & \text{if } 2 \not| n_3, \end{cases}$$

and, for $\underline{n} \in P_1 \cap \mathbf{Z}^4$,

$$\overline{D}_{\underline{n}} - \Delta_{\underline{n}} = \begin{cases} 0 & \text{if } 2|n_1, 2|n_3, \\ \frac{1}{2}E_4 & \text{if } 2 \not| n_1, 2 \not| n_3, \\ \frac{1}{2}E_2 & \text{if } 2 \not| n_1, 2|n_3, \\ \frac{1}{2}E_2 + \frac{1}{2}E_4 & \text{if } 2|n_1, 2 \not| n_3. \end{cases}$$

This shows that $\overline{D}_{\underline{n}} - \Delta_{\underline{n}}$ is not a periodic function on $\sigma_{\{2,4\}}$ in the sense of Definition 6.2, i.e., in Theorem 4.9 we cannot take $\mathcal{P}$ to be equal to Σ.

5. Singularities with semi-abelian Pic^0

The following result is a direct consequence of Lang's conjecture, proven by McQuillan in [21]. Recall that a semi-abelian variety is a commutative algebraic group such that there is an exact sequence

$$0 \to \mathbf{G}_m^a \to G \to A \to 0,$$

where A is an abelian variety, and $a \in \mathbf{N}$.

Proposition 5.1. *Suppose that G is a not necessarily connected commutative algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic 0, such that G is an extension of a finite abelian group by a semi-abelian variety, H is a finitely generated abelian group, and $\pi : H \to G(\mathbf{k})$ is a group homomorphism. Suppose that $Y \subset G$ is a closed integral subvariety such that $Y(\mathbf{k}) \cap \pi(H)$ is Zariski dense in Y. Then there exist a subgroup M of H and $n_0 \in H$ such that*

$$\pi(n) \in Y(\mathbf{k}) \textit{ iff } n \in n_0 + M.$$

Proof. First suppose that G is a semi-abelian variety. Let $\Gamma = \pi(H)$, and

$$\overline{\Gamma} = \{x \in G(\mathbf{k}) \mid nx \in \Gamma \text{ for some } n \in \mathbf{N}\}.$$

$Y(\mathbf{k}) \cap \overline{\Gamma}$ is Zariski dense in X, so by Lang's conjecture (proven in [21]) $Y = b + \Delta$ for some $b \in G(\mathbf{k})$ and some semi-abelian subvariety Δ of Γ. By assumption, there exists an $n_0 \in H$ such that $\pi(n_0) \in b + \Delta$, so we can assume that $b = \pi(n_0)$. Let

$$M = \{n \in H \mid \pi(n) \in \Delta(k)\},$$

a subgroup of H. We have

$$n_0 + M = \{n \in H \mid \pi(n) \in Y(\mathbf{k})\}.$$

In the case where G is not connected, we have an exact sequence

$$0 \to A \to G \to F \to 0,$$

where F is a finite abelian group and A is a semi-abelian variety. There exists $m_0 \in H$ such that $y_0 = \pi(m_0) \in Y(\mathbf{k})$. Thus $Y' = Y - y_0 \subset A$. Suppose that $Z' \subset A$ is a closed subvariety containing $\pi(H) \cap Y'(\mathbf{k})$. Then $\pi(H) \cap Y(\mathbf{k}) \subseteq Z'(\mathbf{k}) + y_0$. By our assumption, $Y \subset Z' + y_0$, which implies that $Y' \subset Z'$. Thus $\pi(H) \cap Y'(\mathbf{k})$ is Zariski dense in $Y'(\mathbf{k})$.

Let $H' = \pi^{-1}(A(\mathbf{k}))$, a subgroup of H. By the first part of the proof, there exist a subgroup M of H' and $m_1 \in H'$ such that $\pi(n) \in Y'(\mathbf{k})$ if and only if $n \in m_1 + M$. Thus $\pi(n) \in Y(\mathbf{k})$ if and only if $\pi(n - m_0) \in Y'(\mathbf{k})$, which holds if and only if $n - m_0 \in m_1 + M$, and this is true if and only if $n \in (m_0 + m_1) + M$. □

The conclusion of Proposition 5.1 is false even for $\mathbf{G}_m$ ([18]) and for abelian varieties ([9], Example 3) over a field of characteristic $p > 0$.

Example 5.2. *The conclusion of Proposition 5.1 is false for arbitrary commutative algebraic groups over an algebraically closed field* $\mathbf{k}$ *of characteristic zero.*

Proof. We first observe that the only nontrivial integral closed subgroups G of $\mathbf{G}_a^2$ over $\mathbf{C}$ are the lines through the origin. This can be seen easily. Choose $0 \neq x \in G(\mathbf{C})$. Let H be the line through the origin containing x. Then H is a closed subgroup; thus $H \cap G$ is a closed subgroup containing x. Since $\sharp\langle x\rangle = \infty$, we have $\sharp(H \cap G) = \infty$. Since both H and G have dimension 1, we have $H = H \cap G = G$. In particular, we see that a translation of an integral nontrivial subgroup of $\mathbf{G}_a^2$ is defined by the vanishing of a linear equation

$$ax + by + c = 0.$$

If Y is an integral curve and $\sharp(Y \cap \mathbf{Z}^2) = \infty$, then the Zariski closure of $Y \cap \mathbf{Z}^2$ in Y is Y. Thus, if Y is not a line, it is not the translation of a semi-abelian subvariety of $\mathbf{G}_a^2$. In particular, as suggested in [29], Y can be taken to be the integral curve in $\mathbf{G}_a^2$ defined by Pell's equation $y^2 - 2x^2 = 1$. □

Given a proper $\mathbf{k}$-scheme Z over an algebraically closed field $\mathbf{k}$, let $\mathrm{Pic}^\tau(Z)$ and $\mathrm{Pic}^0(Z)$ be the subsets of $\mathrm{Pic}(Z)$ of invertible sheaves on Z which are numerically equivalent to 0 and algebraically equivalent to 0 respectively. By the theory of the Picard scheme developed in [11] and [24] (cf. [9], section 2) there exists a group scheme Pic^τ_Z such that $\mathrm{Pic}^\tau_Z(\mathbf{k}) \cong \mathrm{Pic}^\tau(Z)$. If Pic^0_Z is the connected component of the identity of Pic^τ_Z, then $\mathrm{Pic}^0_Z(\mathbf{k}) \cong \mathrm{Pic}^0(Z)$ and $\mathrm{Pic}^\tau(Z)/\mathrm{Pic}^0(Z)$ is a finite group (Theorem 4 of [20], Proposition 14 of [9]). If Y is another proper $\mathbf{k}$-scheme and $\phi : Z \to Y$ is a morphism, ϕ induces a morphism of group schemes $\mathrm{Pic}^\tau_Y \to \mathrm{Pic}^\tau_Z$, hence also $\mathrm{Pic}^0_Y \to \mathrm{Pic}^0_Z$, such that $\mathrm{Pic}^\tau_Y(\mathbf{k}) \to \mathrm{Pic}^\tau_Z(\mathbf{k})$ is the pullback homomorphism.

Let $f : X \to \mathrm{spec}(R)$ and $E_1, \ldots, E_r$ be as in section 4. Suppose that $\mathbf{k}$ is algebraically closed of characteristic zero. Let

$$\mathrm{Pic}^0(X) = \{\mathcal{L} \in \mathrm{Pic}(X) \mid (\mathcal{L} \cdot E_i) = 0 \text{ for } 1 \leq i \leq r\}.$$

Lemma 5.3. *Let D be an effective divisor on X with exceptional support. Then, the induced morphism of algebraic groups* $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(D)$ *is surjective, and* $\mathrm{Pic}^0(D) = \mathrm{Pic}^\tau(D)$.

Proof. $\mathrm{Pic}^0(D) = \mathrm{Pic}^\tau(D)$, since D is a curve. (In our case, it follows from the analysis of [2] that the kernel of the degree map $d : \mathrm{Pic}(D) \to \mathbf{Z}^r$ is connected. Since this kernel is $\mathrm{Pic}^\tau(D)$, we have $\mathrm{Pic}^\tau(D) = \mathrm{Pic}^0(D)$.) $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ is surjective by Lemma 14.3 of [19]. Thus if $\mathcal{L} \in \mathrm{Pic}^0(D)$, there exists $\mathcal{M} \in \mathrm{Pic}(X)$ such that $\mathcal{M} \otimes \mathcal{O}_D \cong \mathcal{L}$. Let $E_{i_1}, \dots, E_{i_m}$ be the integral exceptional curves which are not in the support of D. Let $r_{i_j} = (\mathcal{M} \cdot E_{i_j})$ for $1 \le j \le m$. Since R is Henselian, there exist integral curves D_{i_j}, $1 \le j \le m$, which are disjoint from D and E_{i_k} if $k \ne j$, and such that $(D_{i_j} \cdot E_{i_k}) = 1$. Let $\mathcal{N} = \mathcal{M} \otimes \mathcal{O}_X(-\sum_{j=1}^m r_{i_j} D_{i_j})$. $\mathcal{N} \in \mathrm{Pic}^0(X)$ and $\mathcal{N} \otimes \mathcal{O}_D \cong \mathcal{L}$. ☐

Lemma 5.4. *Suppose that D_0 is an effective exceptional divisor on X. Then, there exists an effective exceptional divisor D on X such that $D_0 \le D$ and $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(D)$ is an isomorphism.*

Proof. Suppose that F is an effective exceptional divisor with exceptional support such that $-F$ is ample and $D_0 \le F$. Then

$$H^1(X, \mathcal{O}_X^*) \cong \varprojlim H^1(X, \mathcal{O}_{nF}^*).$$

There exists $n_0 > 0$ such that $H^1(X, \mathcal{O}_F(-nF)) = 0$ for $n \ge n_0$. The proof of Lemma 1.4 in [2] implies

$$H^1(X, \mathcal{O}_{(n+1)F}^*) \cong H^1(X, \mathcal{O}_{nF}^*)$$

for $n \ge n_0$. Thus

$$H^1(X, \mathcal{O}_X^*) \cong H^1(X, \mathcal{O}_{nF}^*). \qquad ☐$$

In the proof of the next lemma we will use Chevalley's theorem (cf. Proposition 11 in Chapter III of [26]) which tells us that if G is a commutative algebraic group, then there is an exact sequence of algebraic groups

$$0 \to L \to G \to B \to 0,$$

where B is an abelian variety and L is a (commutative) linear algebraic group. Further, $L \cong \mathbf{G}_m^\beta \times \mathbf{G}_a^\gamma$ (cf. [26], Proposition 12 in Chapter III and the corollary to Proposition 8 in Chapter VIII).

Theorem 5.5. *Suppose that $\mathrm{Pic}^0(X)$ is a semi-abelian variety, and D is an effective divisor with exceptional support. Then $\mathrm{Pic}^0(D)$ is semi-abelian.*

Proof. The pullback morphism $\Lambda : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(D)$ is the morphism of $\mathbf{k}$-valued points induced by a surjective morphism of algebraic groups (by Lemmas 5.3 and 5.4). By assumption, $\mathrm{Pic}^0(X)$ is a semi-abelian variety, so there is an exact sequence of algebraic groups

$$0 \to \mathbf{G}_m^\alpha \to \mathrm{Pic}^0(X) \to A \to 0,$$

where A is an abelian variety. There is an exact sequence

$$0 \to L \to \mathrm{Pic}^0(D) \to B \to 0,$$

where B is an abelian variety and L is a commutative linear algebraic group. We have $\Lambda(\mathbf{G}_m^\alpha) \subset L$, since a rational map of $\mathbf{P}^1$ to an abelian variety is trivial (cf. the corollary to Lemma 7 in Chapter III of [26]). Suppose that $\Lambda(\mathbf{G}_m^\alpha) \ne L$. We can form the quotient $Z = \mathrm{Pic}^0(D)/\Lambda(\mathbf{G}_m^\alpha)$, which is as algebraic group (cf. Chapter II of [3]).

We have a surjective morphism $\overline{\Lambda} : A \to Z$. There is an extension

$$0 \to L_1 \to Z \to C \to 0,$$

where C is an abelian variety and L_1 is a nontrivial commutative linear algebraic group. Then $\overline{\Lambda}^{-1}(L_1)$ is a subgroup of A. Since A contains no rational curves, the connected component B of $\overline{\Lambda}^{-1}(L_1)$ containing the identity is an abelian variety which surjects onto L_1. We have an inclusion

$$\Gamma(L_1, \mathcal{O}_{L_1}) \subset \Gamma(B, \mathcal{O}_B) = \mathbf{k}.$$

Since L_1 is affine, $L_1 = \mathrm{spec}(\mathbf{k})$.

Thus we have a surjection

$$\Lambda : \mathbf{G}_m^{\alpha} \to L \cong \mathbf{G}_m^{\beta} \times \mathbf{G}_a^{\gamma}$$

for some $\beta, \gamma \geq 0$. Suppose that $\gamma > 0$. Taking an inclusion and a quotient, we have a surjective homomorphism of algebraic groups, $\phi : \mathbf{G}_m \to \mathbf{G}_a$. Every root of unity in $\mathbf{G}_m(\mathbf{k}) \cong \mathbf{k}^*$ must map to 0 in $\mathbf{G}_a(\mathbf{k}) \cong \mathbf{k}$. Thus the kernel K of ϕ, which is a closed subgroup of $\mathbf{G}_m$, is infinite. Thus $K = \mathbf{G}_m$, and we have a contradiction, showing that $\gamma = 0$. Thus $\mathrm{Pic}^0(D)$ is a semi-abelian variety. □

Proposition 5.6. *Suppose that the reduced exceptional locus E of X is a simple normal crossing divisor. Then $\mathrm{Pic}^0(X)$ is a semi-abelian variety if and only if $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(E)$ is an isomorphism.*

Proof. Since E is a reduced divisor with normal crossings, it follows from [2], p. 488, that there is an exact sequence

$$0 \to \mathbf{G}_m^N \to \mathrm{Pic}^0(E) \to \prod_{i=1}^{r} \mathrm{Pic}^0(E_i) \to 0$$

for some $N \in \mathbf{N}$. Since each E_i is a smooth projective curve, each $\mathrm{Pic}^0(E_i)$ is an abelian variety. We thus see that (with our assumptions on E) $\mathrm{Pic}^0(E)$ is a semi-abelian variety, and the sufficiency of the condition follows.

Let D be an effective exceptional divisor such that $\mathrm{Pic}^0(X) \cong \mathrm{Pic}^0(D)$ and $D \geq E$, as in Lemma 5.3. The analysis of [2] shows that there is a surjection $\mathrm{Pic}^0(D) \to \mathrm{Pic}^0(E)$, and the kernel has a composition series with factors isomorphic to $\mathbf{G}_a$, and thus is isomorphic to $\mathbf{G}_a^{\alpha}$ for some α. If $\mathrm{Pic}^0(X)$ is a semi-abelian variety, we must then have that the kernel is trivial. □

6. Structure of the Hilbert function $h : \mathbf{N}^r \to \mathbf{N}$

Let R be a complete normal local ring of dimension two, and let $f : X \to \mathrm{spec}(R)$ be a resolution of singularities with integral exceptional divisors $E_1, \ldots, E_r$. Recall that $h(\underline{n}) = \ell\left(R/H^0(X, \mathcal{O}_X(-D_{\underline{n}}))\right)$ for $\underline{n} \in \mathbf{N}^r$. In this section we will prove a structure theorem for the Hilbert-Samuel function h.

The following result will play an analogous role to Lemma 4.8.

Lemma 6.1. *Let S be a subset of $\{1, \ldots, r\}$ and $J = \{1, \ldots, r\} \setminus S$. For any subset T of J and any map $\alpha : T \to \frac{1}{e_S}\mathbf{N}$, there exist a map $\alpha^h : J \setminus T \to \frac{1}{e_S}\mathbf{N}$ and an effective divisor C_α with support in $\bigcup_{i \in S \cup T} E_i$ such that:*

(i) $\overline{D_{\underline{n}}} - \Delta_{\underline{n}} = \overline{D_{\underline{m}}} - \Delta_{\underline{m}}$ *for* $\underline{n}, \underline{m} \in P(\alpha, \alpha^h) \cap \mathbf{Z}^r$, $\underline{n} - \underline{m} \in e_S\mathbf{Z}^r$.

(ii) *For* $\underline{n} \in P(\alpha, \alpha^h) \cap \mathbf{Z}^r$*, we have*

$$H^1\left(X, \mathcal{O}_X(-\overline{D_{\underline{n}}})\right) = H^1\left(X, \mathcal{O}_{C_\alpha}(-\overline{D_{\underline{n}}})\right).$$

In particular, if $S \cup T = \emptyset$ *then* $H^1\left(X, \mathcal{O}_X(-\overline{D_{\underline{n}}})\right) = 0$ *for* $\underline{n} \in P(\alpha, \alpha^h) \cap \mathbf{Z}^r$*.*

Proof. Let $\alpha^c : J \setminus T \to \frac{1}{e_S}\mathbf{N}$ be the map obtained in Lemma 4.8. Then, for $\underline{n} \in P(\alpha, \alpha^c) \cap \mathbf{Z}^r$, the $\mathbf{Q}$-divisor $\overline{D_{\underline{n}}} - \Delta_{\underline{n}}$ reaches only a finite number of values $B_1, \ldots, B_t$. Hence, for every $\underline{n} \in P(\alpha, \alpha^c) \cap \mathbf{Z}^r$, there exists k, $1 \leq k \leq t$, such that

$$\overline{D_{\underline{n}}} = \sum_{j \in J \setminus T} l_j^S(\underline{n}) \Delta_j + \sum_{j \in T} \alpha(j) \Delta_j + B_k.$$

Let F be such that $-F$ is ample. From the exact sequence (8) tensored with $\mathcal{O}_X(-\overline{D}_{\underline{n}})$ (and since $-\overline{D}_{\underline{n}}$ is nef) it follows that there exists $m_0 \in \mathbf{N}$ such that

$$H^1\left(X, \mathcal{O}_X(-\overline{D}_{\underline{n}})\right) = H^1\left(X, \mathcal{O}_{m_0 F}(-\overline{D}_{\underline{n}})\right) \quad \text{for all } \underline{n} \in P(\alpha, \alpha^c) \cap \mathbf{Z}^r.$$

In an analogous way as in the proof of Lemma 2.7, let $m_0 F = C_\alpha + C'_\alpha$, where $\operatorname{Supp} C_\alpha \subseteq \bigcup_{i \in S \bigcup T} E_i$ and $\operatorname{Supp} C'_\alpha \subseteq \bigcup_{j \in J \setminus T} E_j$. Then, there exists $b \in \mathbf{N}$ such that, if $\underline{n} \in P(\alpha, \alpha^c) \cap \mathbf{Z}^r$ and $l_j^S(\underline{n}) \geq b$ for $j \in J \setminus T$, then $H^1(X, \mathcal{O}_{C'_\alpha}(-C_\alpha - \overline{D}_{\underline{n}})) = 0$; hence

$$H^1\left(X, \mathcal{O}_{m_0 F}(-\overline{D_{\underline{n}}})\right) = H^1\left(X, \mathcal{O}_{C_\alpha}(-\overline{D_{\underline{n}}})\right).$$

Therefore, the map $\alpha^h : J \setminus T \to \frac{1}{e_S}\mathbf{N}$ given by

$$\alpha^h(j) = \sup\{\alpha^c(j), b\} \quad \text{for } j \in J \setminus T$$

satisfies the result. $\square$

Definition 6.2. A function $\varphi : \mathbf{Z}^r \to \mathbf{Q}$ is called *periodic* if there exists a subgroup H of $\mathbf{Z}^r$ such that $\mathbf{Z}^r / H$ is finite and

$$\varphi(\underline{n}) = \varphi(\underline{m}) \quad \text{for } \underline{n}, \underline{m} \in \mathbf{Z}^r, \underline{n} - \underline{m} \in H.$$

For example, from (28) it follows that, for each polyhedral set P in the abstract complex $\mathcal{P}$ in Theorem 4.9, each of the components of the function

$$P \cap \mathbf{Z}^r \to \mathbf{E}_{\mathbf{Q}} \cong \mathbf{Q}^r, \quad \underline{n} \mapsto \overline{D_{\underline{n}}} - \Delta_{\underline{n}}$$

can be extended to a periodic function on $\mathbf{Z}^r$.

Proposition 6.3. *There exists an abstract complex of polyhedral sets* $\mathcal{P}$ *subdividing the fan* Σ *in Corollary 4.3 with the same associated cones, such that, for* $\underline{n} \in \mathbf{N}^r$*,*

$$h(\underline{n}) = Q(\underline{n}) + L(\underline{n}) + \varphi(\underline{n}),$$

where, for each $P \in \mathcal{P}$*, we have:*

(i) *For* $\underline{n} \in P \cap \mathbf{N}^r$*,* $Q(\underline{n})$ *is equal to a polynomial of degree two with coefficients in* $\mathbf{Q}$*.*

(ii) *For* $\underline{n} \in P \cap \mathbf{N}^r$ *we have* $L(\underline{n}) = \sum_{i=1}^r \varphi_i(\underline{n})\, n_i$*, where* φ_i *is equal to a periodic function, for* $1 \leq i \leq r$*.*

(iii) φ *is bounded.*

More precisely, if $P \subseteq \sigma_S$ *(there always exists such an* S*), then*

(i′) *the polynomial in (i) has coefficients in* $\frac{1}{2(e_S)^2}\mathbf{Z}$*, and is the same for all* $P' \in \mathcal{P}$ *such that* $P' \subseteq \sigma_S$*; and* (ii′) *the functions* φ_i *in (ii) satisfy*

$$\varphi_i(\underline{n}) = \varphi_i(\underline{m}) \quad \textit{for } \underline{n}, \underline{m} \in P \cap \mathbf{Z}^r, \underline{n} - \underline{m} \in e_S\mathbf{Z}^r.$$

Proof. From the Riemann-Roch formula (6),

$$\begin{aligned} h(\underline{n}) &= \ell(R/H^0(X, \mathcal{O}_X(-\overline{D}_{\underline{n}}))) \\ &= \tfrac{1}{2}(-(K_X \cdot \overline{D}_{\underline{n}}) - (\overline{D}_{\underline{n}})^2) + h^1(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X(-\overline{D}_{\underline{n}})) \\ &= \tfrac{1}{2}(-(K_X \cdot \Delta_{\underline{n}}) - (\Delta_{\underline{n}})^2) + \tfrac{1}{2}(-K_X \cdot (\overline{D}_{\underline{n}} - \Delta_{\underline{n}}) - (\overline{D}_{\underline{n}} - \Delta_{\underline{n}})^2) \\ &\quad -((\overline{D}_{\underline{n}} - \Delta_{\underline{n}}) \cdot \Delta_{\underline{n}}) + h^1(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X(-\overline{D}_{\underline{n}})). \end{aligned}$$

If Σ is the fan in Corollary 4.3, and we take $\mathcal{P}$ to be the abstract complex of polyhedral sets in Theorem 4.9, which subdivides Σ with the same associated cones, then, from Corollary 4.3 and Theorem 4.9 it follows that

$$h(\underline{n}) = Q(\underline{n}) + L(\underline{n}) + \varphi_0(\underline{n}) - h^1(X, \mathcal{O}_X(-\overline{D_{\underline{n}}})),$$

where, for each $P \in \mathcal{P}$, Q (resp. L) satisfies (i) and (i′) (resp. (ii) and (ii′)), and φ_0 also satisfies (ii′).

The function $\underline{n} \mapsto h^1(X, \mathcal{O}_X(-\overline{D_{\underline{n}}}))$ is bounded by Lemma 2.4. □

Corollary 6.4. *There exists an abstract complex of polyhedral sets* $\mathcal{P}$ *subdividing the fan* Σ *in Corollary 4.3 with the same associated cones such that, for each* $P \in \mathcal{P}$, *there exist* $m_P \in \mathbf{N}$, *a set* $\{Q_{P,i}(\underline{n})\}_{i=1}^{m_P}$ *of polynomials of degree two, and a function* $\gamma_P : P \cap \mathbf{Z}^r \to \{1, \ldots, m_P\}$ *such that*

$$h(\underline{n}) = Q_{P,\gamma_P(\underline{n})}(\underline{n}) \quad \textit{for } \underline{n} \in P \cap \mathbf{Z}^r.$$

Definition 6.5. Suppose that $P \subset \mathbf{Q}^r$ is a polyhedral set. We say that a function $\varphi : P \cap \mathbf{Z}^r \to \mathbf{Q}$ is semi-periodic if

1. $I = \text{Image } \varphi$ is a finite set.
2. For each $i \in I$, there exists a finite set Ω_i, and for each $l \in \Omega_i$ we associate an element $\underline{m}_l \in \mathbf{Z}^r$ and a subgroup $\Delta_l \subseteq \mathbf{Z}^r$ such that for $\underline{n} \in P \cap \mathbf{Z}^r$,

$$\varphi(\underline{n}) \geq i \Leftrightarrow \underline{n} \in \bigcup_{l \in \Omega_i} (\underline{m}_l + \Delta_l).$$

Note that periodic implies semi-periodic, and the sum and product of semi-periodic functions are periodic.

Theorem 6.6. *Suppose that the residue field* $\mathbf{k}$ *of* R *is an algebraically closed field of characteristic* 0 *and, either* $r \leq 2$, *or* $r > 2$ *and* $\text{Pic}^0(X)$ *is a semi-abelian variety. Then, there exists an abstract complex of polyhedral sets* $\mathcal{P}$ *subdividing the fan* Σ *in Corollary 4.3 with the same associated cones, such that, for* $\underline{n} \in \mathbf{N}^r$,

$$h(\underline{n}) = Q(\underline{n}) + L(\underline{n}) + \varphi(\underline{n}),$$

where, for each $P \in \mathcal{P}$, *we have:*

- (i) *For* $\underline{n} \in P \cap \mathbf{Z}^r$, $Q(\underline{n})$ *is equal to a polynomial of degree two with coefficients in* $\mathbf{Q}$.
- (ii) *For* $\underline{n} \in P \cap \mathbf{Z}^r$ *we have* $L(\underline{n}) = \sum_{i=1}^r \varphi_i(\underline{n})\, n_i$, *where* φ_i *is equal to a periodic function for* $1 \leq i \leq r$.
- (iii) *For* $\underline{n} \in P \cap \mathbf{Z}^r$, $\varphi(\underline{n})$ *is equal to a semi-periodic function.*

Proof. Let Σ be the fan of Corollary 4.3, and $\mathcal{P}$ the abstract complex of polyhedral sets in Theorem 4.9 (and Proposition 6.3) which subdivides Σ with the same associated cones.

By Lemmas 6.1 and 4.6, the abstract complex $\mathcal{P}$ can be refined to an abstract complex of polyhedral sets $\mathcal{P}'$ subdividing Σ with the same associated cones and such that, for every $P \in \mathcal{P}'$, there exist disjoint subsets S, T of $\{1, \ldots, r\}$ and a map $\alpha : T \to \frac{1}{e_S}\mathbf{N}$ such that $P \subseteq P_S(\alpha, \alpha^h)$, where α^h is the map assigned to α in Lemma 6.1.

We will prove that for $P \in \mathcal{P}'$, the function $\underline{n} \mapsto h^1(X, \mathcal{O}_X(-\overline{D_{\underline{n}}}))$ is a semi-periodic function on $P \cap \mathbf{Z}^r$.

Fix a polyhedral set $P \in \mathcal{P}'$. There are S and $\alpha : T \to \frac{1}{e_S}\mathbf{N}$ such that $P \subseteq P(\alpha, \alpha^h)$. If $S \cup T = \emptyset$, then $h^1(X, \mathcal{O}_X(-\overline{D_{\underline{n}}})) = 0$ for $\underline{n} \in P \cap \mathbf{Z}^r$. So, suppose that $S \bigcup T \neq \emptyset$. Then there exists an effective divisor C_α with support in $\bigcup_{i \in S \cup T} E_i$ such that (i) and (ii) in Lemma 6.1 hold. Let $\{\underline{n}_1, \ldots, \underline{n}_t\} \subseteq P \cap \mathbf{Z}^r$ be such that $\underline{n}_i - \underline{n}_k \notin e_S\mathbf{Z}^r$ for $i \neq k$, and for all $\underline{n} \in P \cap \mathbf{Z}^r$ there exists $\underline{n}_k$ such that $\underline{n} - \underline{n}_k \in e_S\mathbf{Z}^r$, i.e.

$$P \cap \mathbf{Z}^r = \bigcup_{k=1}^{t} P \cap (\underline{n}_k + e_S\mathbf{Z}^r),$$

and the union is disjoint. Let H be the subgroup of $e_S\mathbf{Z}^r$ given by the intersection with $e_S\mathbf{Z}^r$ of all hyperplanes $L(\underline{m}) = 0$, where L is an integral linear form such that there exists $b \in \mathbf{Z}$ with $P \subseteq \{\underline{n} \in \mathbf{Q}^r \mid L(\underline{n}) = b\}$. Then

$$P \cap \mathbf{Z}^r = \bigcup_{k=1}^{t} P \cap (\underline{n}_k + H), \tag{29}$$

and the union is disjoint. Note that $H \subseteq \{\underline{m} \in \mathbf{Q} \mid l_j^S(\underline{m}) = 0 \quad \forall j \in T\}$, since $P \subseteq P(\alpha, \alpha^h)$. Thus, for $\underline{n} \in P \cap (\underline{n}_k + H)$,

$$\overline{D_{\underline{n}}} = \Delta_{\underline{n}} + (\overline{D_{\underline{n}_k}} - \Delta_{\underline{n}_k}) = \Delta^S_{\underline{n}-\underline{n}_k} + \overline{D_{\underline{n}_k}} = \sum_{j \notin S \cup T} l_j^S(\underline{n} - \underline{n}_k)\Delta_j + \overline{D_{\underline{n}_k}}.$$

Therefore, for

$$\underline{m} \in (-\underline{n}_k + P) \cap H = P \cap (\underline{n}_k + H) - \underline{n}_k,$$

$\sum_{j \notin S \cup T} l_j^S(\underline{m})\Delta_j$ is a divisor. We can, if necessary, replace H with the subgroup spanned by $\bigcup_k(-\underline{n}_k + P) \cap H$. Thus, for all $\underline{m} \in H$, $\sum_{j \notin S \cup T} l_j^S(\underline{m})\Delta_j$ is a divisor. Then, since $\operatorname{Supp} C_\alpha \subseteq \bigcup_{i \in S \cup T} E_i$, we have a group homomorphism

$$\pi : H \to \operatorname{Pic}^0(C_\alpha), \qquad \underline{m} \mapsto \sum_{j \notin S \cup T} l_j^S(\underline{m})\ \Delta_j \cdot C_\alpha, \tag{30}$$

such that

$$\mathcal{O}_{C_\alpha}(\overline{D_{\underline{n}}}) = \pi(\underline{n} - \underline{n}_k) + \mathcal{O}_{C_\alpha}(\overline{D_{\underline{n}_k}}) \quad \text{for } \underline{n} \in P \cap (\underline{n}_k + H). \tag{31}$$

For fixed k, $1 \leq k \leq t$, the sets

(32)

$$\Omega_{k,i} = \{\mathcal{L} \in \operatorname{Pic}^0(C_\alpha) \mid h^1(C, \mathcal{L}^{-1} \otimes \mathcal{O}_{C_\alpha}(-\overline{D_{\underline{n}_k}})) \geq i\}, \quad \text{for } i \in \mathbf{N},$$

are closed sets of $\operatorname{Pic}^0(C_\alpha)$ that define a chain

$$\ldots \Omega_{k,i+1} \subseteq \Omega_{k,i} \subseteq \ldots \subset \Omega_{k,0} = \operatorname{Pic}^0(C_\alpha)$$

(cf. the proof of Theorem 8 in [9]). The chain is stationary, so there exists ω_k such that $\Omega_{k,i} = \Omega_{k,\omega_k}$ for $i \geq \omega_k$.

By (ii) in Lemma 6.1, (31) and (32), we have

$$h^1\left(X, \mathcal{O}_X(-\overline{D_{\underline{n}}})\right) \geq i \iff \pi(\underline{n} - \underline{n}_k) \in \Omega_{k,i} \quad \text{for } \underline{n} \in P \cap (\underline{n}_k + H). \tag{33}$$

Suppose first that $r \leq 2$. Since $S \cup T \neq \emptyset$, then $\sharp(\{1, \ldots, r\} \setminus (S \cup T)) \leq 1$, and the semi-periodicity follows as in Theorem 2.8 ([9], Theorem 8).

Now suppose that $r > 2$; thus $\mathrm{Pic}^0(X)$ is a semi-abelian variety. By Theorem 5.5, $\mathrm{Pic}^0(C_\alpha)$ is also a semi-abelian variety. For fixed k, $1 \leq k \leq t$, let us consider the group homomorphism $\pi : H \to \mathrm{Pic}^0(C_\alpha)$ in (30) and, for $1 \leq i \leq l_k$, the closed set $\Omega_{k,i}$ of $\mathrm{Pic}^0(C_\alpha)$ in (32). From Proposition 5.1 applied to π and each of the irreducible components of the Zariski closure in $\mathrm{Pic}^0(C_\alpha)$ of $\Omega_{k,i} \cap \pi(H)$, we conclude that there exist a finite set $\Delta_{k,i}$ and, for each $\delta \in \Delta_{k,i}$, $\underline{m}_\delta \in H$ and a subgroup M_δ of H such that

$$\pi(\underline{m}) \in \Omega_{k,i} \iff \underline{m} \in \bigcup_{\delta \in \Delta_{k,i}} (\underline{m}_\delta + M_\delta).$$

Let $\psi_{k,i} : H \to \mathbf{N}$ be defined by

$$\psi_{k,i}(\underline{m}) = \begin{cases} 1 & \text{if } \underline{m} \in \bigcup_{\delta \in \Delta_{k,i}} (\underline{m}_\delta + M_\delta), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi_k = \sum_{i=1}^{\omega_k} \psi_{k,i}$, and

$$\psi : \bigcup_{k=1}^{t} (\underline{n}_k + H) \to \mathbf{N}, \quad \psi(\underline{n}) = \psi_k(\underline{n} - \underline{n}_k) \ \text{ if } \underline{n} \in (\underline{n}_k + H),$$

which is well defined since the union in (29) is disjoint. By (33) we have

$$h^1(X, \mathcal{O}_X(-\overline{D_{\underline{n}}})) = \psi(\underline{n}) \quad \text{for } \underline{n} \in P \cap \mathbf{Z}^r.$$

Hence the result follows. $\square$

7. Rationality of the series defined by h

In this section we will derive from Theorem 6.6 the rationality of the series $\sum_{\underline{n} \in \mathbf{N}^r} h(\underline{n})\, \underline{t}^{\underline{n}}$ when the conditions of Theorem 6.6 hold (Theorem 7.7).

A cone σ is strongly convex if $\{0\}$ is the maximal linear subspace contained in σ. A polyhedral set P in $\mathbf{Q}^r$ is a module over its associated cone σ_P in the sense that $\sigma_P + P \subset P$.

Theorem 7.1. *Let P be a polyhedral set in $\mathbf{Q}^r$ whose associated cone σ_P is strongly convex. Then $P \cap \mathbf{Z}^r$ is a finitely generated module over the semigroup $\sigma_P \cap \mathbf{Z}^r$.*

Let $P = \{\underline{n} \in \mathbf{Q}^r \mid L_i(\underline{n}) \geq b_i, 1 \leq i \leq m\}$, where $m \in \mathbf{N}$ and, for $1 \leq i \leq m$, L_i is an integral linear form on $\mathbf{Q}^r$ and $\underline{b} = (b_1, \ldots, b_m) \in \mathbf{Z}^r$, as in (21). Define $L : \mathbf{Z}^r \to \mathbf{Z}^m$ by $\underline{v} \to (L_1(\underline{v}), \ldots, L_m(\underline{v}))$. L is 1-1, since σ_P contains only the trivial linear subspace. Let $D_+, D_- \subset \mathbf{Q}^m$ be the regions

$$D_+ = \{(x_1, \ldots, x_m) \mid x_i \geq 0, 1 \leq i \leq m\},$$

$$D_- = \{(x_1, \ldots, x_m) \mid x_i \leq 0, 1 \leq i \leq m\}.$$

Set $\overline{P} = P \cap \mathbf{Z}^r$, $\overline{\sigma_P} = \sigma_P \cap \mathbf{Z}^r$. L gives 1-1 correspondences between $\overline{\sigma_P}$ and $\mathrm{Image}(L) \cap D_+$ and between $\overline{P}$ and $\mathrm{Image}(L) \cap (\underline{b} + D_+)$. There is a partial order on $\mathbf{Q}^m$, $\underline{v}_1 \leq \underline{v}_2$ if all components of $\underline{v}_1$ are $\leq$ the corresponding components of $\underline{v}_2$.

Definition 7.2. We say that $\underline{v} \in \overline{P}$ is a minimal generator if $\underline{v} = \underline{w} + \underline{v}_0$ with $\underline{w} \in \overline{P}$, $\underline{v}_0 \in \overline{\sigma_P}$ implies $\underline{v} = \underline{w}$.

Theorem 7.1 is an immediate consequence of the following lemma:

Lemma 7.3. (i) *The set of minimal generators of* $\overline{P}$ *generates* $\overline{P}$ *as a* $\overline{\sigma_P}$*-module.*

(ii) *If* $\underline{v}$ *is a minimal generator of* $\overline{P}$*, then*

$$(L(\underline{v}) + D_-) \cap L(\overline{P}) = \{L(\underline{v})\}.$$

(iii) *There are only finitely many minimal generators.*

Proof. We first prove (i). Suppose that $\underline{w} \in \overline{P}$. If $\underline{w}$ is a minimal generator, we are done. Else, there exists $\underline{w}_1 \in \overline{P}$ and $\underline{v}_1 \in \overline{\sigma_P}$ such that $\underline{v}_1 \neq 0$ and $\underline{w} = \underline{w}_1 + \underline{v}_1$.

If $\underline{w}_1$ is not a minimal generator, we can repeat. In this way either we realize $\underline{w}$ as a sum $\underline{w} = \underline{w}' + \underline{v}'$ with $\underline{w}'$ a minimal generator, $\underline{v}' \in \overline{\sigma_P}$, or we construct an infinite sequence

$$\underline{w} = \underline{w}_1 + \underline{v}_1, \underline{w}_1 = \underline{w}_2 + \underline{v}_2, \cdots, \underline{w}_i = \underline{w}_{i+1} + \underline{v}_{i+1}, \cdots$$

with $\underline{0} \neq \underline{v}_i \in \overline{\sigma_P}$ and $\underline{w}_i \in \overline{P}$ for all i. We have then an infinite sequence of integral vectors

$$L(\underline{w}) > L(\underline{w}_1) > \cdots > L(\underline{w}_i) > \cdots$$

in the region $\underline{b} + D_+$, a contradiction.

Now we prove (ii). Suppose that $\underline{v}$ is a minimal generator which does not satisfy this property. Set $\underline{w} = L(\underline{v})$. Then there exists $\underline{w} \neq \underline{w}_1 \in L(\overline{P}) \cap (\underline{w} + D_-)$. There is $\underline{v}_1 \in \overline{P}$ such that $\underline{w}_1 = L(\underline{v}_1)$. Also,

$$L(\underline{v} - \underline{v}_1) = \underline{w} - \underline{w}_1 \in D_+ \cap \mathrm{Image}(L)$$

implies $\underline{v} - \underline{v}_1 \in \overline{\sigma_P}$, and $\underline{v} = \underline{v}_1 + (\underline{v} - \underline{v}_1)$ implies $\underline{v}$ is not a minimal generator, a contradiction.

Finally, we prove (iii). Let $\{\underline{v}_i\}_{i \in I}$ be the set of minimal generators of $\overline{P}$ over $\overline{\sigma_P}$. To each minimal generator $\underline{v}_i$, associate

$$\underline{u}_i = L(\underline{v}_i) - \underline{b} \in D_+ \cap \mathbf{Z}^m = \mathbf{N}^m.$$

The $\underline{u}_i$ are pairwise not comparable by our partial order by (ii). Let J be the ideal

$$J = (\underline{x}^{\underline{u}_i} \mid i \in I) \subset k[x_1, \ldots, x_m].$$

J is finitely generated and $\{\underline{x}^{\underline{u}_i} \mid i \in I\}$ is a minimal set of generators of J, so I is finite. $\square$

Corollary 7.4. *Let* P *be a polyhedral set in* $\mathbf{Q}^r$ *whose associated cone* σ_P *is strongly convex. Then, there exist a polynomial* $p(t_1, \ldots, t_r) \in \mathbf{Z}[t_1, \ldots, t_r]$*,* $s \in \mathbf{N}$*, and nonzero* $\underline{a}_1, \ldots, \underline{a}_s \in \mathbf{N}^r$ *such that*

$$\sum_{\underline{n} \in P \cap \mathbf{Z}^r} \underline{t}^{\underline{n}} = \underline{t}^{\underline{b}} \, \frac{p(t_1, \ldots, t_r)}{\prod_{i=1}^{s} (1 - \underline{t}^{\underline{a}_i})}.$$

Proof. If $\mathbf{k}$ is a field, then

$$\mathbf{k}[\sigma_P \cap \mathbf{Z}^r] = \bigoplus_{\underline{n} \in \sigma_P \cap \mathbf{Z}^r} \mathbf{k}\ \underline{t}^{\underline{n}}$$

is a $\mathbf{N}^r$-graded $\mathbf{k}$-algebra, which is finitely generated over $\mathbf{k}$ ([28], Theorem I.3.1). By Theorem 7.1,

$$\mathbf{k}[P \cap \mathbf{Z}^r] = \bigoplus_{\underline{n} \in P \cap \mathbf{Z}^r} \mathbf{k}\ \underline{t}^{\underline{n}}$$

is a finitely generated $\mathbf{Z}^r$-graded $k[\sigma_P \cap \mathbf{Z}^r]$-module. Its Hilbert series is

$$\sum_{\underline{\alpha} \in \mathbf{Z}^r} \dim \mathbf{k}[P \cap \mathbf{Z}^r]_{\underline{\alpha}}\ \underline{t}^{\underline{\alpha}},$$

and

$$\dim \mathbf{k}[P \cap \mathbf{Z}^r]_{\underline{\alpha}} = \begin{cases} 1 & \text{if } \underline{\alpha} \in P \cap \mathbf{Z}^r, \\ 0 & \text{otherwise.} \end{cases}$$

By [28], Theorem I.2.3, we conclude the result. $\square$

Theorem 7.5. *Let $P \subseteq \mathbf{Q}^r_{\geq 0}$ be a polyhedral set in $\mathbf{Q}^r$. Let $M \subseteq \mathbf{Z}^r$ be a subgroup and $\underline{m} \in \mathbf{Z}^r$. Then, there exist a polynomial $p(t_1, \ldots, t_r) \in \mathbf{Z}[t_1, \ldots, t_r]$, $s \in \mathbf{N}$, and nonzero $\underline{a}_1, \ldots, \underline{a}_s \in \mathbf{N}^r$ such that*

$$\sum_{\underline{n} \in P \cap (\underline{m} + M)} \underline{t}^{\underline{n}} = \frac{p(t_1, \ldots, t_r)}{\prod_{i=1}^s (1 - \underline{t}^{\underline{a}_i})}.$$

Proof. We have

$$P \cap (\underline{m} + M) = \underline{m} + ((-\underline{m} + P) \cap M)$$

if the intersection is nonempty. The set $-\underline{m} + P$ is a polyhedral set in $\mathbf{Q}^r$. Write

$$M = \bigoplus_{j=1}^{r'} \mathbf{Z}\ \underline{m}_j$$

for some $r' \leq r$. Set

$$M_{\mathbf{Q}} = M \otimes_{\mathbf{Z}} \mathbf{Q} = \bigoplus_{j=1}^{r'} \mathbf{Q}\ \underline{m}_j \subseteq \mathbf{Q}^r$$

and $P' = (-\underline{m} + P) \cap M_{\mathbf{Q}}$. With respect to the basis $\{\underline{m}_1, \ldots, \underline{m}_{r'}\}$, we can identify M with $\mathbf{Z}^{r'}$ and $M_{\mathbf{Q}}$ with $\mathbf{Q}^{r'}$. Then, P' is a polyhedral set in $\mathbf{Q}^{r'}$ whose associated cone is strongly convex. Corollary 7.4 implies

$$\sum_{(\lambda_1, \ldots, \lambda_{r'}) \in P' \cap \mathbf{Z}^{r'}} T_1^{\lambda_1} \ldots T_{r'}^{\lambda_{r'}} = \underline{T}^{\underline{\beta}}\ \frac{p'(T_1, \ldots, T_r)}{\prod_{i=1}^s (1 - \underline{T}^{\underline{\alpha}_i})}$$

for some $p'(T_1, \ldots, T_r) \in \mathbf{Z}[T_1, \ldots, T_{r'}]$, where the $\underline{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,r'}) \in \mathbf{Z}^{r'}$, $1 \leq i \leq s$, are such that $\{\sum_{j=1}^{r'} \alpha_{i,j} \underline{m}_j\}_{i=1}^s$ generate $\sigma_P \cap M$ as $\mathbf{k}$-algebra. Since

$$\sum_{\underline{n} \in P \cap (\underline{m} + M)} \underline{t}^{\underline{n}} = \underline{t}^{\underline{m}} \sum_{(\lambda_1, \ldots \lambda_{r'}) \in P' \cap \mathbf{Z}^{r'}} (\underline{t}^{\underline{m}_1})^{\lambda_1} \ldots (\underline{t}^{\underline{m}_{r'}})^{\lambda_{r'}},$$

we conclude that

$$\sum_{\underline{n}\in P\cap(\underline{m}+M)} \underline{t}^{\underline{n}} = \frac{\underline{t}^{\underline{c}}\, p(t_1,\ldots,t_r)}{\underline{t}^{\underline{d}}\prod_{i=1}^{s}(\underline{t}^{\underline{a}_i}-\underline{t}^{\underline{b}_i})},$$

where $\underline{t}^{\underline{c}}\, p(t_1,\ldots,t_r)$ and $\underline{t}^{\underline{d}}\prod(\underline{t}^{\underline{a}_i}-\underline{t}^{\underline{b}_i})$ are relatively prime in $\mathbf{Q}[t_1,\ldots,t_r]$, and $\underline{b}_i-\underline{a}_i=\sum_{j=1}^{r'}\alpha_{i,j}\underline{m}_j$. Since $P\subseteq\mathbf{Q}^r_{\geq 0}$,

$$\sum_{\underline{n}\in P\cap(\underline{m}+M)} \underline{t}^{\underline{n}} \in \mathbf{Q}[[t_1,\ldots,t_r]]\cap\mathbf{Q}(t_1,\ldots,t_r)=\mathbf{Q}[t_1,\ldots,t_r]_{(t_1,\ldots,t_r)}.$$

Thus $\underline{d}=0$, and $\underline{a}_i=0$ or $\underline{b}_i=0$ for all i. $\square$

By formal differentiation of both sides of 7.5, we obtain the following corollary.

Corollary 7.6. *Let $P\subseteq\mathbf{Q}^r_{\geq 0}$ be a polyhedral set in $\mathbf{Q}^r$. Let $\underline{m}\in\mathbf{Z}^r$, let $M<\mathbf{Z}^r$ be a subgroup, and suppose that $q(\underline{n})$ is a polynomial in $\mathbf{Q}[t_1,\ldots,t_r]$. Then, there exist $s\in\mathbf{N}$, nonzero $\underline{a}_1,\ldots,\underline{a}_s\in\mathbf{N}^r$, $d_i\in\mathbf{N}$ and a polynomial $p(t_1,\ldots,t_r)\in\mathbf{Q}[t_1,\ldots,t_r]$ such that*

$$\sum_{\underline{n}\in P\cap(\underline{m}+M)} q(\underline{n})\underline{t}^{\underline{n}} = \frac{p(t_1,\ldots,t_r)}{\prod_{i=1}^{s}(1-\underline{t}^{\underline{a}_i})^{d_i}}$$

with $d_i\leq\deg(q)+1$.

Theorem 7.7. *Suppose that $\mathbf{k}=R/m$ is an algebraically closed field of characteristic 0, and either $r\leq 2$, or $r>2$ and $\mathrm{Pic}^0(X)$ is a semi-abelian variety. Then, there exist $s\in\mathbf{N}$, nonzero $\underline{a}_1,\ldots,\underline{a}_s\in\mathbf{N}^r$, $d_i\in\mathbf{N}$ and a polynomial $p(t_1,\ldots,t_r)\in\mathbf{Q}[t_1,\ldots,t_r]$ such that*

$$\sum_{\underline{n}\in\mathbf{N}^r} h(\underline{n})\ \underline{t}^{\underline{n}} = \frac{p(t_1,\ldots,t_r)}{\prod_{i=1}^{s}(1-\underline{t}^{\underline{a}_i})^{d_i}}.$$

In particular, the series

$$\sum_{\underline{n}\in\mathbf{N}^r} h(\underline{n})\ \underline{t}^{\underline{n}}$$

is a rational series.

Proof. Let notation be as in Theorem 6.6 and its proof. Let Σ be the fan consisting of the cones σ_S of Definition 4.1 and their faces, and let $\mathcal{P}$ be the abstract complex of polyhedral sets subdividing Σ of Theorem 6.6. Put a well ordering on $\mathcal{P}$. Let $\alpha=\#(\mathcal{P})$. Then

$$\sum_{\underline{n}\in\mathbf{N}^r} h(\underline{n})\underline{t}^{\underline{n}} = \sum_{i=1}^{\alpha}(-1)^{i-1}\left(\sum_{\gamma_1<\cdots<\gamma_i}\left(\sum_{\underline{n}\in P_{\gamma_1}\cap\cdots\cap P_{\gamma_i}\cap\mathbf{Z}^r} h(\underline{n})\underline{t}^{\underline{n}}\right)\right).$$

We are reduced to showing that if $a>0$ and $\gamma_1<\cdots<\gamma_a$, then

$$\sum_{\underline{n}\in P_{\gamma_1}\cap\cdots\cap P_{\gamma_a}\cap\mathbf{Z}^r} h(\underline{n})\underline{t}^{\underline{n}}$$

is rational. Let $\Lambda=P_{\gamma_1}\cap\cdots\cap P_{\gamma_a}$, which is a polyhedral set. Let $P=P_{\gamma_1}$. By assumption there exists $\sigma_S\in\Sigma$ such that $P\subset\sigma_S$. With the notation of the proof of Theorem 6.6, we have $\{\underline{n}_1, \ldots, \underline{n}_{\overline{t}}\}$ such that $P \cap \mathbf{Z}^r$ is the disjoint union

$$P \cap \mathbf{Z}^r = \bigcup_{k=1}^{\overline{t}} P \cap (\underline{n}_k + e_S \mathbf{Z}^r).$$

Thus

$$\sum_{\underline{n} \in \Lambda \cap \mathbf{Z}^r} h(\underline{n}) t^{\underline{n}} = \sum_{k=1}^{\overline{t}} \sum_{\underline{n} \in \Lambda \cap \{\underline{n}_k + e_S \mathbf{Z}^r\}} h(\underline{n}) t^{\underline{n}}.$$

We can now fix k with $1 \leq k \leq \overline{t}$. For $\underline{n} \in \Lambda \cap \{\underline{n}_k + e_S \mathbf{Z}^r\}$, we have

$$h(\underline{n}) = q(\underline{n}) + \psi(\underline{n}),$$

where $q(\underline{n})$ is a quadratic polynomial and $\psi(\underline{n}) = h^1(X, \mathcal{O}_X(-\overline{D}_{\underline{n}}))$ is a semi-periodic function. We have

$$\sum_{\underline{n} \in \Lambda \cap \{\underline{n}_k + e_S \mathbf{Z}^r\}} h(\underline{n}) t^{\underline{n}} = \sum_{\underline{n} \in \Lambda \cap \{\underline{n}_k + e_S \mathbf{Z}^r\}} q(\underline{n}) t^{\underline{n}} + \sum_{\underline{n} \in \Lambda \cap \{\underline{n}_k + e_S \mathbf{Z}^r\}} \psi(\underline{n}) t^{\underline{n}}.$$

By corollary 7.6, the first series is rational.

Let $\Delta_{k,i}$ be the sets of the proof of Theorem 6.6, with associated $\underline{m}_\delta \in \mathbf{Z}^r$ and subgroup M_δ of $\mathbf{Z}^r$ for $\delta \in \Delta_{k,i}$. Let $\omega = \max \psi(\underline{n})$. Let $B_k = \Lambda \cap \{\underline{n}_k + e_S \mathbf{Z}^r\}$. For $\underline{n} \in B_k$ we have

$$\psi(\underline{n}) \geq i \Leftrightarrow \underline{n} \in \bigcup_{\delta \in \Delta_{k,i}} (\underline{m}_\delta + M_\delta)$$

and

$$\sum_{\underline{n} \in B_k} \psi(\underline{n}) t^{\underline{n}} = \sum_{i=1}^{\omega} \left(\sum_{\underline{n} \in B_k} \chi_i(\underline{n}) t^{\underline{n}} \right),$$

where

$$\chi_i(\underline{n}) = \begin{cases} 1 & \text{if } \underline{n} \in \bigcup_{\delta \in \Delta_{k,i}} (\underline{m}_\delta + M_\delta), \\ 0 & \text{if } \underline{n} \notin \bigcup_{\delta \in \Delta_{k,i}} (\underline{m}_\delta + M_\delta). \end{cases}$$

Fix i and well order the set $\Delta_{k,i}$. Then

$$\sum_{\underline{n} \in B_k} \chi_i(\underline{n}) t^{\underline{n}} = \sum_{j \geq 1} (-1)^{j-1} \left(\sum_{\delta_1 < \cdots < \delta_j} \left(\sum_{\underline{n} \in B_k \cap (\underline{n}_{\delta_1} + M_{\delta_1}) \cap \cdots \cap (\underline{n}_{\delta_j} + M_{\delta_j})} t^{\underline{n}} \right) \right),$$

where $\delta_1, \ldots, \delta_j$ are in $\Delta_{k,i}$. If $\underline{n}_0 \in (\underline{n}_{\delta_1} + M_{\delta_1}) \cap \cdots \cap (\underline{n}_{\delta_j} + M_{\delta_j})$, then

$$(\underline{n}_{\delta_1} + M_{\delta_1}) \cap \cdots \cap (\underline{n}_{\delta_j} + M_{\delta_j}) = \underline{n}_0 + M_{\delta_1} \cap \cdots \cap M_{\delta_j}.$$

Thus rationality now follows from Theorem 7.5. $\square$

Corollary 7.8. *Suppose that the divisor class group* $\mathrm{Cl}(R)$ *of* R *is an extension of a finite group by a semi-abelian variety, and* $f : X \to \mathrm{spec}(R)$ *is a resolution of singularites. Then the Poincaré series*

$$\sum_{\underline{n} \in \mathbf{N}^r} h(\underline{n}) \underline{t}^{\underline{n}}$$

is a rational series.

Proof. $\mathrm{Cl}(R)$ is an extension of a finite group by $\mathrm{Pic}^0(X)$ [19]. The result is then immediate from Theorem 7.7. □

8. Irrationality in characteristic $p > 0$

In this section we construct an example where the residue field has positive characteristic $p > 0$, there are two irreducible exceptional components, and the associated series are not rational. Recall that if the residue field has characteristic zero, and there are two exceptional components, then the associated series must be rational (Theorem 7.7).

Theorem 8.1. *There exists a two-dimensional complete normal local ring R containing a field of characteristic $p > 0$, and a resolution of singularities $g : W \to \mathrm{spec}(R)$ with two exceptional divisors C_0 and D such that the series*

$$\sum_{m,n\in\mathbf{N}} h^1(W, \mathcal{O}_W(-mC_0 - nD))s^m t^n \tag{34}$$

and

$$\sum_{m,n\in\mathbf{N}} \ell(R/\Gamma(W, \mathcal{O}_W(-mC_0 - nD)))s^m t^n \tag{35}$$

are not rational series.

Throughout this section we will use the notation of Example 5 from [9]. In this example, a two-dimensional complete normal local ring R is constructed which contains a field of characteristic $p > 0$, and a resolution of singularities $g : W \to \mathrm{spec}(R)$ with two exceptional divisors C_0 and D. Example 5 from [9] gives an explicit calculation of $h^1(W, \mathcal{O}_W(-nC_0 - nD))$ as a function of n. This function is bounded, but is not eventually periodic.

We will analyze this example to give counterexamples to the rationality questions which we consider in this paper, in positive characteristic.

Proposition 8.2. *Suppose that $0 < b \le a \le 3b - 3$. Then*

$$h^1(W, \mathcal{O}_W(-aC_0 - bD)) = \begin{cases} 0 & \textit{if } a - b \ge 2, \\ 0 & \textit{if } a - b = 1, a \textit{ is not a power of } p, \\ 1 & \textit{if } a - b = 1, a \textit{ is a power of } p, \\ 1 & \textit{if } a = b, a+1 \textit{ is not a power of } p, \\ 2 & \textit{if } a = b, a+1 \textit{ is a power of } p. \end{cases}$$

Proof. The case $a = b$ is proven in Example 5 of [9]. The same arguments extend to prove the theorem. In fact, in the case $a > b$, we also reduce to

$$H^1(W, \mathcal{O}_W(-aC_0 - bD)) \cong H^1(W, \mathcal{O}_{2C_0}(-aC_0 - bD)).$$

In

$$\begin{aligned} &0 \to \mathcal{O}_C((a+1)(\eta - P) + (a+1-b)P) \to \mathcal{O}_{2C_0}(-aC_0 - bD) \\ &\quad \to \mathcal{O}_C(a(\eta - P) + (a-b)P) \to 0 \end{aligned} \tag{36}$$

$a \geq b+2$ implies all H^1 in (36) vanish. If $a = b+1$, then

$$\begin{aligned}
&h^1(C, \mathcal{O}_C((a+1)(\eta - P) + (a+1-b)P)) = 0,\\
&h^1(C, \mathcal{O}_C(a(\eta - P) + (a-b)P)) = h^1(C, \mathcal{O}_C(a(\eta - P) + P)\\
&\quad = \begin{cases} 0 & \text{if } a \text{ is not a power of } p,\\ 1 & \text{if } a \text{ is a power of } p. \end{cases}
\end{aligned}$$

□

Let $g : W \to \operatorname{spec}(R)$ be the morphism of Example 5 [9]. Set

$$a_{ij} = h^1(W, \mathcal{O}_W(-iC_0 - jD)),$$
$$h = \sum_{i,j \geq 0} a_{ij} s^i t^t.$$

Suppose that h is rational, so that there exists a nonzero polynomial

$$Q = \sum_{i,j=0}^{r} b_{ij} s^i t^j$$

such that Qh is a polynomial,

$$Qh = \sum_{m,n} \left(\sum_{i,j=0}^{r} b_{ij} a_{m-i,n-j} \right) s^m t^n.$$

There exists n_0 such that

$$\sum_{i,j=0}^{r} b_{ij} a_{m-i,n-j} = 0$$

for $m, n \geq n_0$.

Set $m = n + r + 1$, and suppose that $n \geq \max\{n_0 + r, 2r + 3, 2pr\}$. Then

$$\sum_{i,j=0}^{r} b_{ij} a_{n+r+1-i,n-j} = 0$$

We have $0 < n - j \leq n + r + 1 - i \leq 3(n-j) - 3$ for $0 \leq i, j \leq r$, so by Proposition 8.2, all the $a_{n+r+1-i,n-j}$ vanish in this range, except for

$$a_{n+1,n} = \begin{cases} 0 & \text{if } n+1 \text{ is not a power of } p,\\ 1 & \text{if } n+1 \text{ is a power of } p. \end{cases}$$

If we take $n = p^t - 1$ with t sufficiently large, we then get

$$0 = b_{r0} a_{n+1,n} = b_{r0}.$$

Now assume that α is a natural number such that $0 \leq \alpha \leq r-1$ and $b_{ij} = 0$ if $i - j \geq \alpha + 1$. We will prove that $b_{ij} = 0$ if $i - j \geq \alpha$.

Suppose that $n \geq \max\{n_0 + r, 2r + 3, 2pr\}$.

Suppose that $0 \leq i, j \leq r$ and $i - j < \alpha$. Set $a = n + \alpha + 1 - i$, $b = n - j$. We then have

$$a - b = n + \alpha + 1 - i - (n - j) = \alpha + 1 + j - i \geq 2,$$
$$3b - 3 = 3(n-j) - 3 \geq 3n - 3r - 3 \geq n + r \geq n + \alpha + 1 - i = a.$$

By Proposition 8.2, $a_{n+\alpha+1-i,n-j} = 0$ if $i - j < \alpha$. So

$$\begin{aligned} 0 &= \textstyle\sum_{i,j=0}^{r} b_{ij} a_{n+\alpha+1-i,n-j} \\ &= \textstyle\sum_{i-j=\alpha} b_{ij} a_{n+\alpha+1-i,n-j} \\ &= \begin{cases} b_{\alpha,0} & \text{if } n = p^t - 1, \\ b_{\alpha+1,1} & \text{if } n = p^t, \\ & \vdots \\ b_{r,r-\alpha} & \text{if } n = p^t + r - \alpha - 1. \end{cases} \end{aligned}$$

(This is where we use $n \geq 2pr$.) Thus $b_{ij} = 0$ if $i - j \geq \alpha$.

We have proved that $b_{ij} = 0$ if $i - j \geq r$, so by descending induction we conclude that $b_{ij} = 0$ for all i, j, and $Q = 0$, a contradiction.

Let

$$h(m, n) = \ell(R/H^0(W, \mathcal{O}_W(-mC_0 - nD)))$$

be the function of (6). By the local Riemann-Roch formula (6), there exists a quadratic polynomial $A(m, n)$ such that

$$\begin{aligned} h(m, n) &= \ell(R/H^0(W, \mathcal{O}_W(-mC_0 - nD))) \\ &= A(m, n) - h^1(W, \mathcal{O}_W(-mC_0 - nD)) \text{ if } m, n \geq 0. \end{aligned}$$

Thus the series (35)

$$\sum_{m,n \in \mathbf{N}} h(m, n) s^m t^n$$

is not rational.

9. Irrationality in characteristic zero

In this section, we construct an example of a resolution $X \to \operatorname{spec}(R)$ where $\mathbf{k} = R/m$ has characteristic zero, there are 3 exceptional components, and the associated Poincaré series is not rational.

Theorem 9.1. *There exists a two-dimensional complete normal local ring R, containing a field of characteristic 0, and a resolution of singularities $g : X \to \operatorname{spec}(R)$ with three exceptional divisors C, F_1 and F_2 such that the series*

$$\sum_{n_1,n_2,n_3 \in \mathbf{N}} h^1(\mathcal{O}_X(-n_1C - n_2F_1 - n_3F_2)) t_1^{n_1} t_2^{n_2} t_3^{n_3} \tag{37}$$

and

$$\sum_{n_1,n_2,n_3 \in \mathbf{N}} \ell(R/\Gamma(\mathcal{O}_X(-n_1C - n_2F_1 - n_3F_2))) t_1^{n_1} t_2^{n_2} t_3^{n_3} \tag{38}$$

are not rational series.

Lemma 9.2. *Suppose that X is a nonsingular projective surface over $\mathbf{C}$, and $F_1, \ldots, F_r$ are integral analytically irreducible closed curves contained in X such that $(F_i \cdot F_j)$ is negative definite. Let $F = F_1 + \cdots + F_r$. Suppose that F is connected. Then there exist a complete normal local ring $\hat{A}$ with maximal ideal m, and a projective $\hat{A}$ scheme $\pi : Y \to \operatorname{spec}(\hat{A})$ such that π is birational, and an isomorphism away from m, $\pi^{-1}(m)_{\mathrm{red}} \cong F$, and the formal schemes Y_F and X_F are isomorphic.*

Proof. By Grauert's contraction theorem [10] there exist a neighborhood S of F (in the complex topology), a normal analytic space U, and a bimeromorphic map $\lambda : S \to U$ such that F is the exceptional locus of λ. Let $q \in U$ be the point such that $\lambda(F) = q$, $A = \mathcal{O}^h_{U,q}$ with maximal ideal n. Since the F_i are analytically irreducible of dimension 1, there exist an effective divisor D on X whose support is F, an ideal sheaf $\mathcal{J}$ on X such that the support of $\mathcal{O}_X/\mathcal{J}$ is 0-dimensional, and if $\mathcal{I} = \mathcal{O}_X(-D)\mathcal{J}$, then $n\mathcal{O}^h_S = \mathcal{I}\mathcal{O}^h_S$. Let $\hat{A}$ be the n-adic completion of A, $m = n\hat{A}$.

Let $V_1, \ldots, V_t$ be an affine cover of F in X, $R_j = \Gamma(V_j, \mathcal{O}_X)$, $I_j = \Gamma(V_j, \mathcal{I})$ for $1 \le j \le t$. We have compatible $\mathbf{C}$-algebra homomorphisms $A/n^i \to R_j/I^i_j$ for all i and j, defined by the composition

$$A/n^i \to \Gamma(S, \mathcal{O}^h_S/\mathcal{I}^i\mathcal{O}^h_S) = \Gamma(X, \mathcal{O}_X/\mathcal{I}^i) \to \Gamma(V_j, \mathcal{O}_X/\mathcal{I}^i).$$

The middle equality is by [27]. We thus have homomorphisms $\hat{A} \to \hat{R}_j$, where $\hat{R}_j$ is the I_j-adic completion of R_j. Each $\hat{R}_j$ is a domain (since R_j is regular and excellent). We have isomorphisms

$$\hat{A} \cong A[[x_1, \ldots, x_n]]/(x_1 - a_1, \ldots, x_s - a_s)$$

and

$$\hat{R}_j \cong R_j[[x_1, \ldots, x_n]]/(x_1 - a_1, \ldots, x_s - a_s),$$

where $(a_1, \ldots, a_s)$ is a basis of n.

Since R_j is of finite type over $\mathbf{C}$, $\hat{R}_j$ is of finite type over $\hat{A}$. Since F is connected, we have an integral scheme $Y = \bigcup \operatorname{spec}(\hat{R}_j)$ of finite type over $\operatorname{spec}(\hat{A})$, with morphism $\pi : Y \to \operatorname{spec}(\hat{A})$ such that $\mathcal{O}_Y/m^i\mathcal{O}_Y \cong \mathcal{O}_X/\mathcal{I}^i$ for all i. By the valuative criterion for properness, Y is proper over $\operatorname{spec}(\hat{A})$, since F is proper over $\mathbf{C}$, and $\hat{A}$ has dimension 2.

Now we see by Grauert's comparison theorem (cf. Chapter III, Section 3 of [4]), [27] and the theorem on formal functions ([12], III, section 4) that

$$\hat{A} = \varprojlim \Gamma(S, \mathcal{O}^h_S/\mathcal{I}^k\mathcal{O}^h_S) = \varprojlim \Gamma(X, \mathcal{O}_X/\mathcal{I}^k) = \Gamma(Y, \mathcal{O}_Y) = \bigcap \hat{R}_j$$

is integrally closed in the function field $\mathbf{C}(Y)$ of Y. Since Y and $\hat{A}$ have dimension 2, the function field $\mathbf{C}(Y)$ of Y is finite over the quotient field $Q(\hat{A})$, so that $Q(\hat{A}) = \mathbf{C}(Y)$. By Zariski's Main Theorem, $\pi : Y \to \operatorname{spec}(\hat{A})$ is an isomorphism away from m, and $\pi^{-1}(m)_{\mathrm{red}} \cong F$.

Finally, we will show that π is projective. There exists an effective divisor $D = \sum b_iF_i$ with support F such that

$$(-D \cdot F_i) > 0 \text{ for } 1 \le i \le r.$$

Thus there exists n_0 such that $\mathcal{O}_D(-nD)$ is very ample and

$$H^1(D, \mathcal{O}_D(-nD)) = 0 \text{ if } n \ge n_0.$$

From the exact sequence

$$0 \to \mathcal{O}_D(-nD) \to \mathcal{O}_{(n+1)D} \to \mathcal{O}_{nD} \to 0$$

we conclude that

$$H^1(X, \mathcal{O}_Y(-lD)) = \varprojlim H^1(Y, \mathcal{O}_{nD}(-lD)) = 0 \text{ if } l \ge n_0.$$

We thus have a surjection

$$H^0(Y, \mathcal{O}_Y(-nD)) \to H^0(D, \mathcal{O}_D(-nD))$$

if $n \geq l$. In particular, $K_a = H^0(Y, \mathcal{O}_Y(-aD))$ is generated by global sections if $a \geq l$, so that $\bigoplus_{n\geq 0} K_n$ is a finitely generated $\hat{A}$ algebra (cf. Proposition III.3.3.1 in [12]). Thus there exists $t \geq l$ such that $\bigoplus_{n\geq 0} K_{tn}$ is generated in degree 1, so that $K_{tn} = K_t^n$ for all $n \geq 1$. If ν_i is the discrete valuation with valuation ring $\mathcal{O}_{Y,F_i}$, then

$$K_a = \{f \in \hat{A} \mid \nu_i(f) \geq ab_i \text{ for } 1 \leq i \leq r\},$$

so that each K_a is an intersection of valuation ideals, and is thus integrally closed. Thus $\bigoplus_{n\geq 0} K_t^n$ is a normal ring, and $Z = \text{proj}(\bigoplus_{n\geq 0} K_t^n)$ is a normal scheme. Since $K_t\mathcal{O}_Y = \mathcal{O}_Y(-tD)$ is invertible, we have a birational morphism $\tau : Y \to Z$ which is an isomorphism away from F. As $K_t\mathcal{O}_D$ is very ample, τ does not contract any component of F. By Zariski's Main Theorem, τ is an isomorphism. □

Lemma 9.3. *There exists a rational, complex Gorenstein projective curve C which has an isolated singularity $\tilde{p}$, with local ring*

$$\mathcal{O}_{C,\tilde{p}} \cong \mathbf{C}[t^2, t^5]_{(t^2,t^5)},$$

and the following properties:

(1) *C has arithmetic genus $p_a(C) = 2$.*
(2) *If $\mathcal{L}$ is a line bundle on C and $\deg(\mathcal{L}) \geq 8$, then $\mathcal{L}$ is generated by global sections.*
(3) *If $\mathcal{L}$ is a line bundle on C and $\deg(\mathcal{L}) \geq 10$, then $\mathcal{L}$ is very ample.*
(4) *If $\mathcal{L}$ is a line bundle on C of negative degree, there exist a nonsingular projective surface S and an embedding $C \subset S$ such that $C \cdot C \sim \mathcal{L}$.*

Proof. Let $C_0 = V(x_2^3x_1^2 - x_0^5) \subset X_0 = \mathbf{P}^2_{\mathbf{C}}$. C_0 is a rational curve with two singular points, $q_0 = (0:0:1)$ and $q_1 = (0:1:0)$. We will resolve the singularity at q_1. There are regular parameters $y = \frac{x_0}{x_1}$, $z = \frac{x_2}{x_1}$ at q_1, and

$$\mathcal{O}_{C_0,q_1} = (\mathbf{C}[y,z]/z^3 - y^5)_{(y,z)}.$$

Let $\pi_1 : X_1 \to X_0$ be the blowup of q_1. Let C_1 be the strict transform of C_0 on X_1. Let $E_1 = \pi_1^{-1}(q_1)$, $q_2 = E_1 \cap C_1$. Let (y_1, z_1) be the regular parameters at q_2 defined by $y = y_1, z = y_1z_1$. Then $z_1^3 - y_1^2 = 0$ is a local equation of C_1 at q_2. Let $\pi_2 : X_2 \to X_1$ be the blowup of q_2. Let C_2 be the strict transform of C_1 on X_2, and let $E_2 = \pi_1^{-1}(q_2)$ and $q_3 = E_2 \cap C_2$. Let (y_2, z_2) be the regular parameters at q_3 defined by $y_1 = y_2z_2, z_1 = z_2$. Then $z_2 - y_2^2 = 0$ is a local equation of C_2 at q_3, and C_2 is thus nonsingular at q_3. Set $\pi = \pi_1 \circ \pi_2$. Identify E_1 with its strict transform on X_2. q_0 is the only singular point on C_2. We have

$$\pi_1^*(C_0) = C_1 + 3E_1.\ \pi^*(C_0) = C_2 + 5E_2 + 3E_1,$$
$$C_2 \cdot C_2 \sim \pi^*(C_0) \cdot C_2 - 5E_2 \cdot C_2 - 3E_1 \cdot C_2.$$

$C_2 \cdot E_2$ and $C_2 \cdot E_1$ are supported at q_2, $y_2 = 0$ is a local equation of E_1 at q_2, and $z_2 = 0$ is a local equation of E_2 at q_2. Thus $C_2 \cdot E_1 = q_2$, $C_2 \cdot E_2 = 2q_2$, and

$$C_2 \cdot C_2 \sim \pi^*(C_0) \cdot C_2 - 13q_2.$$

Let V be a general quintic curve on X_0. We have

$$V \cdot C_0 = p_1 + \cdots + p_{25},$$

where $p_1, \ldots, p_{25} \in C_0$ are distinct nonsingular points. Also,

$$\pi^*(C_0) \cdot C_2 \sim \pi^*(V) \cdot C_2 = p_1 + \cdots + p_{25}.$$

Thus

$$C_2 \cdot C_2 \sim p_1 + \cdots + p_{25} - 13q_2.$$

Set $D = C_2 \cdot C_2$. We have that $\deg(D) = 12$,

$$\hat{\mathcal{O}}_{C_2,q_0} = \mathbf{C}[[\frac{x_0}{x_2}, \frac{x_1}{x_2}]]/(\frac{x_1}{x_2})^2 - (\frac{x_0}{x_2})^5 \cong \mathbf{C}[[t^2, t^5]] \subset \mathbf{C}[[t]]$$

and $\ell(\mathbf{C}[[t]]/\hat{\mathcal{O}}_{C_2,q_0}) = 2$. Let $C = C_2$.

The arithmetic genus of C is thus

$$p_a(C) = p_a(\mathbf{P}^1) + 2 = 2$$

(cf. Exercise IV 1.8 [14]).

Since C is a local complete intersection, the Riemann-Roch theorem is applicable on C (cf. Exercise IV. 1.9 in [14]). In particular, there is a canonical bundle ω_C on C such that for any line bundle $\mathcal{L}$ on C, $h^1(C, \mathcal{L}) = h^0(C, \omega_C \otimes \mathcal{L}^{-1})$ (Serre duality) and $\chi(\mathcal{L}) = \deg(\mathcal{L}) + 1 - p_a(C)$. We further have $\deg(\omega_C) = 2p_a(C) - 2 = 2$.

C has an affine cover by open sets $U_1 = \mathrm{spec}(\mathbf{C}[t^2, t^5])$ and $U_2 = \mathrm{spec}(\mathbf{C}[\frac{1}{t}])$. Let ∞ be the point on C with maximal ideal $(\frac{1}{t})$. ∞ is the point q_3 on C_2. Let $\tilde{p}$ be the point q_0 on C. Let m be the the ideal sheaf of the point $\tilde{p}$. Then $\Gamma(U_1, m) = (t^2, t^5) \subset \mathbf{C}[t^2, t^5])$. Consider the line bundle $\mathcal{L}_1$ defined by

$$H^0(U_1, \mathcal{L}_1) = \mathbf{C}[t^2, t^5]t^2,$$
$$H^0(U_2, \mathcal{L}_1) = \mathbf{C}[\tfrac{1}{t}],$$

and the line bundle $\mathcal{L}_2$ defined by

$$H^0(U_1, \mathcal{L}_2) = \mathbf{C}[t^2, t^5]t^5,$$
$$H^0(U_2, \mathcal{L}_2) = \mathbf{C}[\tfrac{1}{t}].$$

Multiplying $\mathcal{L}_1$ by $\frac{1}{t^2}$ and $\mathcal{L}_2$ by $\frac{1}{t^5}$, we see that $\mathcal{L}_1 \cong \mathcal{O}_C(-2\infty)$ and $\mathcal{L}_2 \cong \mathcal{O}_C(-5\infty)$. We thus have short exact sequences

$$0 \to \mathcal{K}_1 \to \mathcal{O}_C(-2\infty) \oplus \mathcal{O}_C(-5\infty) \to m \to 0, \tag{39}$$

$$0 \to \mathcal{K}_2 \to \mathcal{O}_C(-4\infty) \oplus \mathcal{O}_C(-7\infty) \to m^2 \to 0 \tag{40}$$

of coherent $\mathcal{O}_C$ modules, for some modules $\mathcal{K}_1$ and $\mathcal{K}_2$.

Let $a \in C$ be a closed point, with ideal sheaf m_a, and let $\mathcal{L}$ be a line bundle on C. From the exact sequence

$$0 \to \mathcal{L}m_a \to \mathcal{L} \to \mathcal{L}/\mathcal{L}m_a \to 0$$

we see that $\mathcal{L}$ is generated by global sections if $H^1(C, \mathcal{L}m_a) = 0$ for all $a \in C$. By Serre duality, we have that if $\mathcal{N}$ is a line bundle of degree > 2, then $H^1(C, \mathcal{N}) = 0$. By (39) we see that $\mathcal{L}$ is generated by global sections if $\deg(\mathcal{L}) \geq 8$.

By Proposition II.7.3 in [14], a line bundle $\mathcal{L}$ on C is very ample if

(1) $\mathcal{L}$ is generated by global sections,
(2) $H^0(C, \mathcal{L}m_a) \to H^0(C, \mathcal{L}m_a/\mathcal{L}m_am_b)$ is surjective for distinct closed points $a, b \in C$, and

(3) $H^0(C,\mathcal{L}m_a)\to H^0(C,\mathcal{L}m_a/\mathcal{L}m_a^2)$ is surjective for every closed point $a\in C$.

From the exact sequences

$$0\to\mathcal{L}m_am_b\to\mathcal{L}m_a\to\mathcal{L}m_a/\mathcal{L}m_am_b\to 0$$

and

$$0\to\mathcal{L}m_a^2\to\mathcal{L}m_a\to\mathcal{L}m_a/\mathcal{L}m_a^2\to 0,$$

Serre duality, (39) and (40), we see that $\mathcal{L}$ is very ample if $\deg(\mathcal{L})\geq 10$.

Suppose that $\mathcal{L}\in\mathrm{Pic}(C_2)$ has degree $-e<0$. Let $r=e+12$. Then $\deg(D-\mathcal{L})=r>10$ implies $D-\mathcal{L}$ is very ample, so by Bertini's theorem,

$$D-\mathcal{L}\sim a_1+\cdots+a_r,$$

where $a_1,\dots,a_r\in C_2$ are distinct nonsingular points in C_2. Let $\lambda: X_3\to X_2$ be the blowup of $a_1,\dots,a_r$. Let $F_i=\lambda^{-1}(a_i)$ for $1\leq i\leq r$. Let $C_3\cong C$ be the strict transform of C_2. Then $\lambda^*(C_2)=C_3+F_1+\cdots+F_r$, and

$$C_3\cdot C_3\sim\lambda^*(C_2)\cdot C_3-a_1-\cdots-a_r\sim D-a_1-\cdots-a_r\sim\mathcal{L}.$$

$\square$

Let C be the curve of Lemma 9.3, with singular point $\tilde{p}$. Let $\pi:\mathbf{P}^1\to C$ be the normalization of C, with function field $\mathbf{C}(\mathbf{P}^1)=\mathbf{C}(t)$, where $t=0$ is a local equation of $q=\pi^{-1}(\tilde{p})$. Let $\infty\in C$ be the point with local equation $\frac{1}{t}=0$. Let

$$\mathrm{Div}^0=\text{ group of Weil divisors of degree 0 on } C-\tilde{p}.$$

$$\mathrm{Pic}^0(C)\cong\mathrm{Div}^0/\sim,$$

where $D_1\sim D_2$ if $D_1-D_2=(f)$ for some $f\in\mathbf{C}(t)$ which is a unit in $\mathcal{O}_{C,\tilde{p}}$ (cf. II.6 in [14]).

Suppose that $D\in\mathrm{Div}^0$. There exists $f_D\in\mathbf{C}(t)$ such that $(f_D)=D$ (divisor computed on $\mathbf{P}^1$), and f_D is unique up to multiplication by a nonzero constant in $\mathbf{C}$. Define $\Lambda:\mathrm{Div}^0\to\mathbf{C}^2$ by

$$\Lambda(D)=\left(\frac{d}{dt}\log(f_D)\mid_{t=0},\frac{d^3}{dt^3}\log(f_D)\mid_{t=0}\right).$$

$\Lambda(D)$ is a well defined group homomorphism.

For $D\in\mathrm{Div}^0$, we have an expansion

$$f_D=\sum_{i=0}^{\infty}a_it^i\in\hat{\mathcal{O}}_{\mathbf{P}^1,q}=\mathbf{C}[[t]],$$

where $a_0\neq 0$. We have $f_D\in\hat{\mathcal{O}}_{C,\tilde{p}}$ if and only if $a_1=a_3=0$. Since $f_D\in\mathbf{C}(t)$ and $\mathcal{O}_{C,\tilde{p}}=\hat{\mathcal{O}}_{C,\tilde{p}}\cap\mathbf{C}(t)$ (cf. Lemma 2 in [1]), $f_D\in\mathcal{O}_{C,\tilde{p}}$ if and only if $a_1=a_3=0$. Since $\Lambda(D)=0$ if and only if $a_1=a_3=0$, we have $\Lambda(D)=0$ if and only if $D\sim 0$. By allowing a_1 and a_3 to vary, we see that Λ is onto. Thus Λ is a group isomorphism of $\mathrm{Pic}^0(C)$ with $\mathbf{C}^2$.

We will consider the Abel-Jacobi map

$$AJ: C-\{\tilde{p}\}\to\mathbf{C}^2$$

defined by $AJ(\alpha)=\Lambda(\alpha-\infty)$. We have

$$AJ(\alpha)=\left(\frac{d\log(t-\alpha)}{dt}\mid_{t=0},\frac{d^3\log(t-\alpha)}{dt^3}\mid_{t=0}\right)=\left(\frac{1}{-\alpha},\frac{2}{-\alpha^3}\right).$$

Define the image of the Abel-Jacobi map to be

$$W = AJ(C - \{\tilde{p}\}).$$

W is the subvariety of $\mathbf{C}^2$ defined by $y = 2x^3$.

Lemma 9.4. *Suppose that $D \in Div^0$. Then*

$$h^1(C, \mathcal{O}_C(D+\infty)) = h^0(C, \mathcal{O}_C(D+\infty)) = \begin{cases} 0 & \text{if } \Lambda(D) \notin W, \\ 1 & \text{if } \Lambda(D) \in W. \end{cases}$$

Proof. There exists $f \in \mathbf{C}(t)$ such that $(f) = D$ (divisor computed on $\mathbf{P}^1$). Then

$$H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(D+\infty))$$

has $\mathbf{C}$ basis $\frac{1}{f}, \frac{t}{f}$, and

$$H^0(C, \mathcal{O}_C(D+\infty)) = \left\{\lambda \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(D+\infty)) \mid \lambda \in \mathcal{O}_{C,\tilde{p}}\right\}.$$

There is an expansion

$$\frac{t}{f} = a_1 t + a_2 t^2 + \cdots$$

in $\mathbf{C}[[t]] = \hat{\mathcal{O}}_{\mathbf{P}^1,q}$, where $a_1 \neq 0$, so $\frac{t}{f} \notin \mathcal{O}_{C,\tilde{p}}$. For $0 \neq \alpha \in \mathbf{C}$,

$$\begin{array}{ll} \frac{t-\alpha}{f} \in \mathcal{O}_{C,\tilde{p}} & \Leftrightarrow \Lambda(-D + \alpha - \infty) = 0 \\ & \Leftrightarrow \Lambda(D) = AJ(\alpha). \end{array}$$

We further have

$$\frac{1}{f} \in \mathcal{O}_{C,\tilde{p}} \Leftrightarrow \Lambda(-D) = 0 \Leftrightarrow \Lambda(D) = AJ(\infty) = 0.$$

Since $p_a(C) = 2$, the equality of h^0 and h^1 follows from the Riemann-Roch theorem. □

Suppose that $\mathcal{M}$ is a line bundle on C of degree $-d$ with $d \geq 3$, and p_1, p_2 are distinct points on $C - \{\tilde{p}\}$. Let $\overline{\mathcal{L}} = \mathcal{M} \otimes \mathcal{O}_C(p_1 + p_2)$. Lemma 9.3 shows that there exists a nonsingular surface S and an embedding $C \subset S$ such that $C \cdot C = \overline{\mathcal{L}}$. Also, $\deg(\overline{\mathcal{L}}) < 0$ and C is a local complete intersection in S, so by Lemma 9.2, there is a birational (projective) morphism $\pi : Y \to \operatorname{spec}(R)$, where R is a complete normal local ring of dimension 2, with exceptional divisor C such that Y is nonsingular, $C \cdot C = \overline{\mathcal{L}}$.

Let $X \to Y$ be the blowup of Y at the points p_1 and p_2, with exceptional divisors F_1 and F_2 (both isomorphic to $\mathbf{P}^1$). Identify C with its strict transform on X. Then $X \to \operatorname{spec}(R)$ is a resolution with exceptional curves F_1, F_2 and C, $C \cdot C \sim \mathcal{M}$, $C \cdot F_1 = p_1$, $C \cdot F_2 = p_2$, $(F_1^2) = -1$, $(F_2^2) = -1$.

Lemma 9.5. *Suppose that $\mathcal{N}$ is a line bundle on X such that*

$$(\mathcal{N} \cdot C) > 4 - d, (\mathcal{N} \cdot F_1) \geq 0, (\mathcal{N} \cdot F_2) \geq 0$$

or

$$(\mathcal{N} \cdot C) = 4 - d, \mathcal{N} \cdot C \not\sim \omega_C + \overline{\mathcal{L}}, (\mathcal{N} \cdot F_1) \geq 0, (\mathcal{N} \cdot F_2) \geq 0.$$

Then $H^1(X, \mathcal{N}) \cong H^1(C, \mathcal{O}_C \otimes \mathcal{N})$.

Proof. We have exact sequences

$$0 \to \mathcal{O}_{F_1+F_2}(-n(F_1+F_2+C)-C)\otimes\mathcal{N} \to \mathcal{O}_{(n+1)(F_1+F_2+C)}\otimes\mathcal{N} \to \mathcal{O}_{n(F_1+F_2+C)+C}\otimes\mathcal{N}\to 0 \tag{41}$$

for $n \geq 0$, and

(42)
$$0 \to \mathcal{O}_C(-n(F_1+F_2+C))\otimes\mathcal{N} \to \mathcal{O}_{n(F_1+F_2+C)+C}\otimes\mathcal{N} \to \mathcal{O}_{n(F_1+F_2+C)}\otimes\mathcal{N}\to 0$$

for $n \geq 1$. Moreover,

$$\begin{aligned}(C\cdot(-n(F_1+F_2+C))+(C\cdot\mathcal{N}) &= n(d-2)+(C\cdot\mathcal{N})\\ &\geq (d-2)+(C\cdot\mathcal{N}) \geq 2p_a(C)-2=2\end{aligned}$$

and

$$(F_i\cdot(-n(F_1+F_2+C)-C))+(\mathcal{N}\cdot F_i) = -1+(\mathcal{N}\cdot F_i)\geq -1$$

for $i=1,2$. Thus

$$H^1(X,\mathcal{O}_{n(F_1+F_2+C)}\otimes\mathcal{N})\cong H^1(C,\mathcal{O}_C\otimes\mathcal{N})$$

for all $n>0$, and

$$H^1(X,\mathcal{N})\cong \lim_{\leftarrow} H^1(X,\mathcal{O}_{n(F_1+F_2+C)})\otimes\mathcal{N})\cong H^1(C,\mathcal{O}_C\otimes\mathcal{N}).$$

□

We will now fix $d=3$, and continue to assume that $\mathcal{M}$ is a line bundle of degree $-d=-3$ on C. Let $\mathcal{L}=\mathcal{M}+3\infty$. Set

$$\begin{aligned}\Lambda_1 &= -2F_1-F_2-C,\\ \Lambda_2 &= -F_1-2F_2-C,\\ \Lambda_3 &= -F_1-F_2-C.\end{aligned}$$

$\Lambda_1,\Lambda_2,\Lambda_3$ generate the integral lattice $\mathbf{Z}^3$ in $\mathbf{R}F_1+\mathbf{R}F_2+\mathbf{R}C$, and if $C=F_3$, then $(\Lambda_i\cdot F_j)=\delta_{ij}$ for $1\leq i,j\leq 3$. Set $\overline{\Delta}_i=\Lambda_i\cdot C$ for $1\leq i\leq 3$. For a point $p\in C-\tilde{p}$, set $\overline{p}=p-\infty$. Then

$$\begin{aligned}\overline{\Delta}_1 &= -2p_1-p_2-\mathcal{M} &\sim -2\overline{p}_1-\overline{p}_2-\mathcal{L}\\ \overline{\Delta}_2 &= -p_1-2p_2-\mathcal{M} &\sim -\overline{p}_1-2\overline{p}_2-\mathcal{L}\end{aligned}$$

Now we will fix p_1, p_2 and $\mathcal{M}$ so that $\overline{p}_1=(0,0)$, $\overline{p}_2=(-1,-2)$, $\mathcal{L}=(1,1)$. Then $\overline{\Delta}_1=(0,1)$, $\overline{\Delta}_2=(1,3)$. By Lemma 9.4, for $m,n\geq 0$

$$h^1(C,\mathcal{O}_C(m\overline{\Delta}_1+n\overline{\Delta}_2+F_1))=\begin{cases}1 & m=2n^3-3n,\\ 0 & \text{otherwise.}\end{cases} \tag{43}$$

Lemma 9.6. *There exists at most one divisor $D=-aF_1-bF_2-cC$ on X such that $D\cdot C\sim\omega_C$.*

Proof. Suppose that $D\cdot C\sim\omega_C$. Then

$$\deg\omega_C = 2p_a(C)-2=2=(D\cdot C)=-a-b+3c,$$

$$\begin{aligned}\omega_C-2\infty &\sim -a\overline{p}_1-b\overline{p}_2-c\mathcal{L}\\ &\sim -b(-1,-2)-c(1,1)=(b-c,2b-c),\end{aligned}$$

which implies that there exists at most one value of (b,c) such that $D\cdot C\sim\omega_C$. Since $a=3c-b-2$, there is at most one value of (a,b,c). □

Lemma 9.7. *There exists a number λ such that $a+b+c>\lambda$ and $a,b,c>0$ imply*

$$h^1(X,\mathcal{O}_X(a\Lambda_1+b\Lambda_2+c\Lambda_3))=\begin{cases}0 & \text{if } c\geq 2,\\ 1 & \text{if } c=1, a=2b^3-3b-1,\\ 0 & \text{if } c=1, a\neq 2b^3-3b-1.\end{cases}$$

Proof. By Lemma 9.6 there exists $\lambda>0$ such that $a,b,c>0$ and $a+b+c>\lambda$ imply

$$(a\Lambda_1+b\Lambda_2+c\Lambda_3)\cdot C\not\sim\omega_C+(F_1+F_2+C)\cdot C\sim\omega_C+\overline{\mathcal{L}}$$

and

$$(a\Lambda_1+b\Lambda_2+c\Lambda_3)\cdot C\not\sim\omega_C.$$

By Lemma 9.5, $a,b>0$, $c\geq 2$ and $a+b+c>\lambda$ imply

$$H^1(X,\mathcal{O}_X(a\Lambda_1+b\Lambda_2+c\Lambda_3))\cong H^1(C,\mathcal{O}_C(a\overline{\Delta}_1+b\overline{\Delta}_2+c\overline{\Delta}_3))=0,$$

since $(C\cdot(a\Lambda_1+b\Lambda_2+c\Lambda_3))\geq 2=2p_a(C)-2$. By Lemma 9.5,

$$\begin{aligned}H^1(X,\mathcal{O}_X(a\Lambda_1+b\Lambda_2+\Lambda_3))&\cong H^1(C,\mathcal{O}_C(a\overline{\Delta}_1+b\overline{\Delta}_2+\overline{\Delta}_3))\\&=H^1(C,\mathcal{O}_C((a+1)\overline{\Delta}_1+b\overline{\Delta}_2+\infty)),\end{aligned}$$

since

$$a\Lambda_1+b\Lambda_2+\Lambda_3=(a+1)\Lambda_1+b\Lambda_2+F_1,$$

and Lemma 9.7 follows from (43). □

We will now give the proof of Theorem 9.1. Set $R=\hat{A}$, and let the notation be as above for the surface X. Set $a_{ijk}=h^1(\mathcal{O}_X(-iF_1-jF_2-kC))$. We will first show that the series

$$f=\sum_{i,j,k=0}^{\infty}a_{ijk}t_1^it_2^jt_3^k$$

of (37) is not rational.

Suppose that f is rational. Then there exists a nonzero polynomial

$$Q=\sum_{i,j,k=0}^{r}b_{ijk}t_1^it_2^jt_3^k$$

such that fQ is a polynomial. Thus there is $\sigma>0$ such that

$$\sum_{i,j,k=0}^{r}a_{l-i,m-j,n-k}b_{i,j,k}=0$$

whenever $l+m+n\geq\sigma$ and $l,m,n\geq r$.

We will prove that $Q=0$, and derive a contradiction to the assumption that f is rational, by induction on $0\leq\alpha$ in the following statement:

If i,j,k are such that $0\leq i,j,k\leq r$
and $(3r+1)+i+j-3k<1+\alpha$,
then $b_{ijk}=0$.

The statement is vacuously true for $\alpha = 0$, so we will assume it to be true for α and prove it for $\alpha + 1$. Set

$$\begin{aligned} l &= 2a + b - 1 + \alpha, \\ m &= a + 2b, \\ n &= a + b + r, \end{aligned}$$

where a, b are abitrary, subject to the conditions

$$a \geq \max\{\sigma, \lambda + 4r\}, \quad b \geq \max\{\sigma, \lambda + 4r\},$$

where λ is the integer of Lemma 9.7. For $0 \leq i, j, k \leq r$, set

$$D_{ijk} = -(l-i)F_1 - (m-j)F_2 - (n-k)C.$$

We have $(D_{ijk} \cdot F_1) \geq 1$, $(D_{ijk} \cdot F_2) \geq 1$ and

$$(D_{ijk} \cdot C) = (3r+1) - \alpha + i + j - 3k.$$

We have (by Lemma 9.7)

$$a_{l-i,m-j,n-k} = 0 \text{ if } (3r+1) - \alpha + i + j - 3k > 1$$

and (by assumption)

$$b_{ijk} = 0 \text{ if } (3r+1) - \alpha + i + j - 3k < 1.$$

Thus

$$0 = \sum_{(3r+1)-\alpha+i+j-3k=1} a_{l-i,m-j,n-k} b_{ijk}.$$

If $(3r+1) - \alpha + i + j - 3k = 1$, we have $((D_{ijk} - F_1) \cdot C) = 0$, so

$$D_{ijk} \sim \beta\Lambda_1 + \gamma\Lambda_2 + F_1$$

with

$$\begin{aligned} \beta &= (l-i+1) - (n-k) = a + \alpha - r - i + k, \\ \gamma &= (m-j) - (n-k) = b - r - j + k. \end{aligned}$$

For fixed (a, b) and (i, j) satisfying $0 \leq i, j \leq r$ and $(3r+1) - \alpha + i + j + 3k = 1$,

$$h^1(X, \mathcal{O}_X(D_{ijk})) = \begin{cases} 0 & \text{if } \beta \neq 2\gamma^3 - 3\gamma - 1, \\ 1 & \text{if } \beta = 2\gamma^3 - 3\gamma - 1, \end{cases}$$

by Lemma 9.7, since

$$D_{ijk} \sim (\beta - 1)\Lambda_1 + \gamma\Lambda_2 + \Lambda_3.$$

We now observe that, given $n > 0$, there exists $m(n) > 0$ such that if $x_0 > m(n)$ is an integer, and $y_0 = 2x_0^3 - 3x_0 - 1$, then

$$([x_0 - n, x_0 + n] \times [y_0 - n, y_0 + n]) \cap \mathbf{Z}^2 \cap \{y = 2x^3 - 3x - 1\} = \{(x_0, y_0)\}.$$

We can thus choose (a, b) so that for any $\overline{i}, \overline{j}, \overline{k}$ such that $0 \leq \overline{i}, \overline{j}, \overline{k} \leq r$ satisfying $3r + 1 - \alpha + \overline{i} + \overline{j} + 3\overline{k} = 1$,

$$0 = \sum_{i,j,k=0}^{r} a_{l-i,m-j,n-k} b_{ijk} = b_{\overline{ijk}}.$$

We thus conclude that the series of (37) is not rational, and by the local Riemann-Roch theorem (6), (38) is not rational.

10. A nontrivial semi-periodic h^1

We give an example showing that h^1 is a nontrivial function.

Let $Z = \mathbf{P}^2$ be two-dimensional complex projective space. Let C be a nonsingular cubic curve in Z.

Fix a point $p_\infty \in C$ as the 0 in the group law on C. For $p \in C$, let $\overline{p}$ be the divisor $p - p_\infty$. We then have a group isomorphism $C \to \mathrm{Pic}^0(C)$ given by $p \mapsto \overline{p}$. Let $p_1, p_2 \in C$ be distinct points such that $\overline{p}_1 = -\overline{p}_2 \neq 0$.

By Bertini's theorem, there exists a cubic curve V in Z such that

$$V \cdot C = Q_1 + Q_2 + \cdots + Q_9,$$

where $Q_1, \ldots, Q_9, p_\infty, p_1, p_2$ are distinct points. Let $\pi : X \to \mathbf{P}^2$ be the blowup of $Q_1, \ldots, Q_9, p_1, p_2, p_\infty$. Let $E_1 = \pi^{-1}(p_1)$, $E_2 = \pi^{-1}(p_2)$, $F_i = \pi^{-1}(Q_i)$, for $1 \leq i \leq 9$, $E_\infty = \pi^{-1}(p_\infty)$. Let $\overline{C}$ be the strict transform of C on X. Then

$$\pi^*(C) \cdot \overline{C} \sim V \cdot C = Q_1 + \cdots + Q_9$$

and $\pi^*(C) = \overline{C} + F_1 + \cdots + F_9 + E_\infty + E_1 + E_2$, so that

$$\overline{C} \cdot \overline{C} \sim -p_\infty - p_1 - p_2.$$

By Lemma 9.2, there exist a complete normal local ring R, and a birational morphism $\lambda : Y \to \mathrm{spec}(R)$, where the reduced exceptional fiber of λ is $\overline{C} + E_1 + E_2$ and the formal completion of Y along $\overline{C} + E_1 + E_2$ is isomorphic to the formal completion of X along $\overline{C} + E_1 + E_2$.

Let $\mathcal{L} = -\overline{C} \cdot \overline{C} - 3p_\infty$. Then

$$\begin{aligned} &\mathcal{L} = \overline{p}_1 + \overline{p}_2 \sim 0,\\ &\mathcal{L} - 2\overline{p}_1 - \overline{p}_2 \sim -\overline{p}_1,\\ &\mathcal{L} - \overline{p}_1 - 2\overline{p}_2 \sim -\overline{p}_2 \sim \overline{p}_1. \end{aligned}$$

Set

$$\begin{aligned} \Lambda_1 &= -\overline{C} - 2E_1 - E_2,\\ \Lambda_2 &= -\overline{C} - E_1 - 2E_2,\\ \Lambda_3 &= -\overline{C} - E_1 - E_2, \end{aligned}$$

$$\begin{aligned} &\Lambda_1 \cdot \overline{C} \sim \mathcal{L} - 2\overline{p}_1 - \overline{p}_2 \sim -\overline{p}_1,\\ &\Lambda_2 \cdot \overline{C} \sim \mathcal{L} - \overline{p}_1 - 2\overline{p}_2 \sim \overline{p}_1, \end{aligned}$$

$$h^1(\overline{C}, \mathcal{O}_{\overline{C}}(a\Lambda_1 + b\Lambda_2)) = h^1(\overline{C}, \mathcal{O}_{\overline{C}}((b-a)\overline{p}_1) = \begin{cases} 1, & a = b,\\ 0, & \text{otherwise}. \end{cases}$$

Suppose that $\mathcal{N}$ is a numerically effective line bundle on Y. We have exact sequences

$$0 \to \mathcal{O}_{\overline{C}}(-n(\overline{C} + F_1 + F_2)) \otimes \mathcal{N} \to \mathcal{O}_{n(\overline{C}+F_1+F_2)+\overline{C}} \otimes \mathcal{N} \to \mathcal{O}_{n(\overline{C}+F_1+F_2)} \otimes \mathcal{N} \to 0$$

for $n \geq 1$ and

$$\begin{aligned} 0 \to \mathcal{O}_{F_1+F_2}(-n(\overline{C} + F_1 + F_2) - \overline{C}) \otimes \mathcal{N} \to \mathcal{O}_{(n+1)(\overline{C}+F_1+F_2)} \otimes \mathcal{N}&\\ \to \mathcal{O}_{n(\overline{C}+F_1+F_2)+\overline{C}} \otimes \mathcal{N} \to 0& \end{aligned}$$

for $n \geq 0$. Since

$$H^1(Y, \mathcal{N}) \cong \lim_{\leftarrow} H^1(Y, \mathcal{O}_{n(\overline{C}+F_1+F_2)} \otimes \mathcal{N}),$$

we have

$$H^1(Y, \mathcal{N}) \cong H^1(\overline{C}, \mathcal{O}_{\overline{C}} \otimes \mathcal{N})$$

from the above exact sequences. Thus

$$h^1(Y, \mathcal{O}_Y(a\Lambda_1 + b\Lambda_2)) = \begin{cases} 1, & a = b, \\ 0, & \text{otherwise.} \end{cases}$$

Thus if $i, j, k > 0$ and $i + j = 3k$, we have

$$h^1(Y, \mathcal{O}_Y(-iE_1 - jE_2 - k\overline{C}) = h^1(Y, \mathcal{O}_Y((i-1)\Lambda_1 + (2k-i)\Lambda_2) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

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